Measuring Welfare Changes With Nonlinear Budget Constraints in Continuous and Discrete Hours Labour Supply Models

John Creedy and Guyonne Kalb*

Abstract

This paper examines the computation of exact welfare measures in the context of labour supply models. It is suggested that the standard method of computing compensating and equivalent variations does not allow sufficiently for the nonlinearity of the budget constraint. An exact method is suggested. The method is applied to contexts in which individuals are allowed to vary their hours continuously and where only a limited number of discrete hours of work are available. Discrete hours models have in recent years been used in view of the substantial econometric advantages when estimating the parameters of direct utility functions.

Melbourne Institute Working Paper 9/01
ISSN 1328-4991
ISBN 0 7340 1513 5

*Department of Economics and Melbourne Institute of Applied Economic and Social Research, The University of Melbourne, Parkville, Victoria 3052, Australia.

1 Introduction

The context of income taxation and labour supply is one in which few attempts have been made to compute welfare changes, in terms of compensating and equivalent variations, despite the large number of applied econometric studies using a variety of approaches. This context presents complications in view of the fact that individuals typically face highly nonlinear budget constraints where the appropriate price, the net wage rate, is endogenous as well as labour supply itself. Where welfare changes are discussed in the literature, the standard approach is simply to adopt a minor modification of the well-known expressions used to obtain welfare changes that have been established in the context of commodity demand studies.[1] This involves a central role for expenditure functions, expressing the minimum expenditure required to achieve a specified utility level for a given set of prices.

The aim of this paper is to examine the computation of exact welfare measures in the special context of labour supply models. It is argued that the standard approach does not sufficiently allow for the precise nonlinearity of the budget constraint facing an individual. A method of computing exact welfare change, allowing for the full (relevant) detail of budget constraints, is suggested and illustrated. This paper considers models in which individuals are allowed to vary their hours of work continuously and a framework in which only a limited number of discrete hours of work are available. Such discrete hours models have in recent years been used in applied work in view of the substantial econometric advantages when estimating the parameters of specified direct utility functions. Estimation of direct utility functions using discrete hours avoids the estimation problem concerning the endogeneity of the net wage in standard continuous hours models and avoids the need to

[1] This includes situations in which there may be quantity constraints, as in the context of rationing of standard commodities, or where there are constraints on consumption of environmental resources. In these cases it is usual to define a modified expenditure function conditional on the rationed levels of consumption. See, for example, Johansson (1987).

solve the first-order conditions for utility maximisation, or even to know the full budget constraint facing each individual.[2] In addition, it may be argued that in practice individuals have a limited choice over the extent to which they can vary their hours.

Section 2 examines welfare changes in models in which individuals are allowed to change hours of work continuously. The modification required when only a discrete number of hours levels are allowed is presented in section 3. The use of a discrete hours framework makes the computation of welfare changes much easier if utility functions are such that explicit expressions for demand and expenditure functions cannot be obtained. The next two sections consider specific utility functions which have been used in empirical studies of labour demand. Section 4 uses the quadratic direct utililty function, while section 5 examines the quadratic translog form.[3] In each case numerical examples are provided. Brief conclusions are in section 6.

2 Continuous Labour Supply

Let h denote the number of hours devoted to labour supply, which may be varied continuously, and let c denote net income (or consumption, with the price index normalised to unity). The direct utility function is written as $U(c, h)$. Hence leisure is $T - h$, where T is the total number of hours available for work and leisure. In general, the tax and transfer system is characterised by a piecewise-linear budget constraint.

Any optimal position can be regarded as being generated by a simple linear constraint of the form:

$$c = wh + \mu \tag{1}$$

[2] For a general discussion of alternative approaches, see Creedy and Duncan (2001).

[3] Most applications of the discrete hours framework have used these forms, although Vlasblom *et al.* (2001) used the CES form, for which the approach discussed here can easily be applied.

In the case of tangency solutions, w and μ represent the appropriate net wage rate and virtual income respectively.[4] In the case of a corner solution, virtual income is defined as the value generated by a linear constraint having a net wage, the virtual wage, equal to the slope of the indifference curve at the kink. An important characteristic of the optimal position is that the net wage and virtual income, as well as the hours worked, are endogenous.

The evaluation of welfare changes requires an expression for the expenditure function, giving the minimum expenditure needed to reach a specified indifference curve at a given net wage rate. This can be written in terms of virtual income, using $\mu(w, U)$ or in terms of full income, using $M(w, U)$.[5] If T denotes the total available number of hours, full income, M, is:

$$M = \mu + wT \tag{2}$$

Hence the expenditure function in terms of full income is:

$$M(w, U) = \mu(w, U) + wT \tag{3}$$

2.1 The Simple Case

Suppose there is a change in taxes and transfers from system 0 to system 1. The various terms in each system are indicated by 0 and 1 subscripts. In terms of virtual incomes, the standard expression for the compensating variation is:

$$CV = \mu(w_1, U_0) - \mu_1 \tag{4}$$

Where w_i is either the virtual or actual net wage rate at the optimum position under policy i, U_i is the maximum utility that can be reached under tax system i, and $\mu_i = \mu(w_i, U_i)$. These changes are illustrated in Figure 1, where

[4] Virtual income is therefore distinct from actual non-wage income.

[5] Virtual income, $\mu(w, U)$, is obtained by first obtaining the indirect utility function. Substitute $c = wh + \mu$ into $U(c, h)$, and then substituting the solution for optimal h, from $\left.\frac{dc}{dh}\right|_U = w$ and $c = wh + \mu$, into U. Then invert the indirect utility function by solving U for μ .

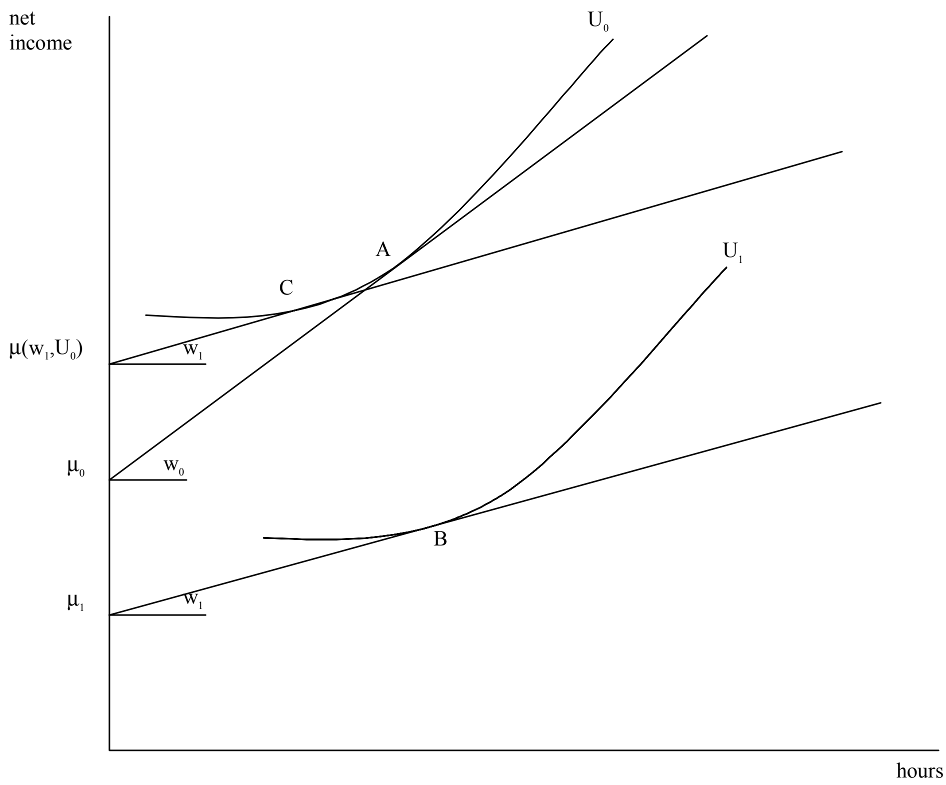

Figure 1: The Simple Case

a tax change involves a movement from point A to point B. The payment of the compensating variation of $\mu(w_1, U_0) - \mu_1$ allows the individual to reach indifference curve U_0 at point C, while in receipt of the net wage w_1 and working fewer hours than at B.

The welfare change can be decomposed into separate price (of leisure) and income effects, using:

$$CV = \{M(w_1, U_0) - M_0\} + (M_0 - M_1) \tag{5}$$

where $M_i = M(w_i, U_i)$. The absolute value of the term in curly brackets corresponds to an area to the left of a compensated leisure demand curve between appropriate prices. A fall in the wage rate, so that $w_1 < w_0$, lowers the price of leisure, so that the welfare change from the price effect, the term in curly brackets in (5) is negative (welfare increases), while reducing full income. Hence the overall effect is ambiguous.

The equivalent variation is the negative of the corresponding compensating variation for a change from tax system 1 to system 0, and is therefore $EV = \mu_0 - \mu(w_0, U_1)$. Since no different principles are involved, much of the following discussion concentrates on compensating variations. However, in later sections, illustrative examples are given for both the CV and EV.

The expressions given above may be described as the 'standard' welfare changes used in the literature. For example, they are used by Hausman (1981, p.672, 1985, pp.243-245), Blomquist (1983, pp.187-190), Blundell *et al.* (1994, pp.4-8), Creedy (2000a, b). However, the approach implicitly assumes that the 'virtual budget line', given by the tangent to U_1 at point B, with associated virtual income of μ_1, does in fact apply over the relevant range. That is, the individual can move to the left of B (and therefore increase consumption of leisure) along the linear budget line until the hours worked correspond to those at point C. The addition of the compensated variation to net income allows consumption to increase so that point C can be reached. But the context of labour supply is one in which the budget constraint facing each individual is highly nonlinear.

2.2 Nonlinear Budget Contraints

The nonlinearity, or piecewise linearity, of budget constraints relating net income to hours worked can imply that the compensating variation, as conventionally defined, would in practice be insufficient to restore the individual to U_0. Consider Figure 2, which shows the relevant ranges of piecewise linear budget constraints before and after a change in the tax structure. The invidual moves from point A on U_0, a tangency solution, to point B on U_1, a corner solution. The standard approach outlined above involves a compensating variation of $\mu(w_1, U_0) - \mu_1$, where in this case μ_1 is the intercept (at

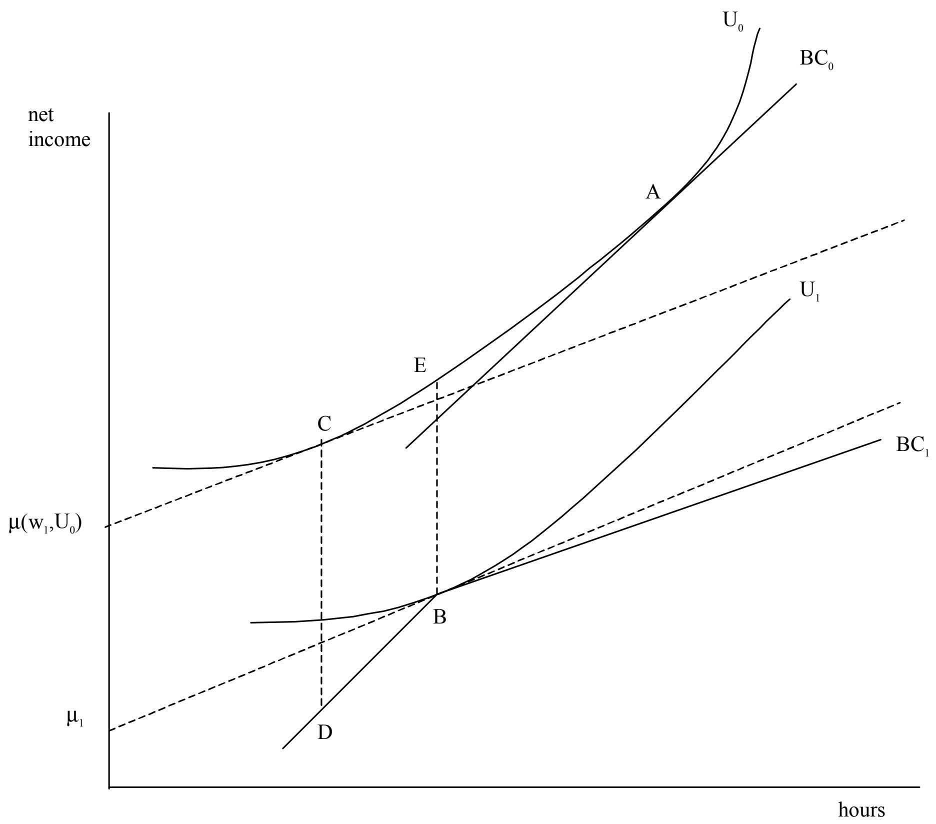

Figure 2: A Nonlinear Constraint

$h = 0$) of a virtual budget constraint with a slope of w_1, rather than the actual constraint.

The ability to vary hours of work continuously means that labour supply can be reduced to correspond to point C, but in this case the increase in leisure involves a fall in net income below that implied by the virtual budget constraint. The vertical distance between D and C is actually required to bring the individual to indifference curve U_0. This is greater than the standard compensating variation as given above. The minimum compensation actually required involves the individual remaining at the hours associated with point B, and is equal to the vertical distance, BE, from B to indifference curve U_0.

This is obtained as follows.[6] Let w_c denote the virtual wage, the slope of the indifference curve U_0, at point E and let $\mu(w_c, U_0)$ represent the associated virtual income. The compensation required is given by the difference between net income at B and net income at E. Hence, if h_1 is the number of hours worked at B and E:

$$CV = \{\mu(w_c, U_0) + w_c h_1\} - \{\mu_1 + w_1 h_1\} \tag{6}$$

which can be rewritten as;

$$CV = \{\mu(w_c, U_0) - \mu_1\} + (w_c - w_1) h_1 \tag{7}$$

or:

$$\begin{aligned} CV &= \{\mu(w_1, U_0) - \mu_1\} \\ &\quad + \{\mu(w_c, U_0) - \mu(w_1, U_0)\} + (w_c - w_1) h_1 \end{aligned} \tag{8}$$

This shows the addition to the conventional value (contained in the first set of curly brackets) that is needed to allow for the fact that leftward movement from B cannot take place along the virtual constraint through point B, but involves a lower net income. In terms of full incomes, the adjusted compensating variation is given by:

$$\begin{aligned} CV &= \{M(w_1, U_0) - M_1\} + \{M_0 - M_1\} \\ &\quad + \{M(w_c, U_0) - M(w_1, U_0)\} + (w_1 - w_c)(T - h_1) \end{aligned} \tag{9}$$

where again the first line gives the standard expression. The equivalent variation is the amount that must be taken from an individual in order to keep utility constant at the new level, if the tax change were reversed. In this example, the standard expression for the equivalent variation is not affected so long as the kinks are far enough from the tangency point.

[6] This assumes that the budget set is convex over the relevant range. Allowance for further complexities is discussed in the following subsection.

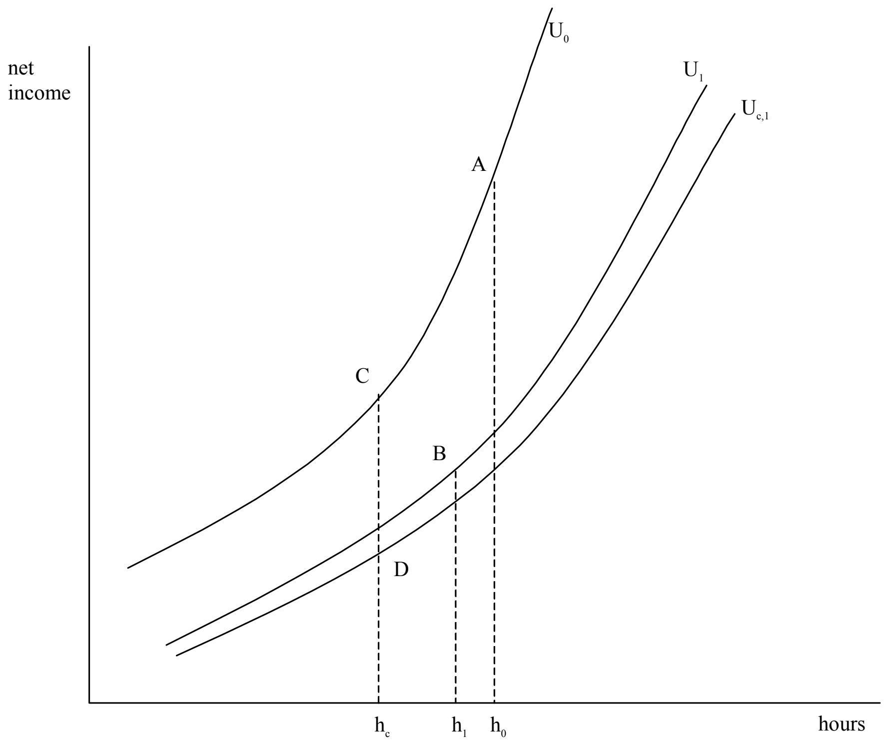

Figure 3: The General Case

2.3 The General Case

The previous subsection was concerned with a case where the relevant range of the budget constraint has one kink. In general, allowance must be made for more complex types of constraint. For example, the constraint to the left of point B in Figure 2 may have a number of convex and concave ranges over a relatively small range of hours of work.[7] It is quite possible (even if B were a tangency solution) that there is a level of hours, say h_c, where the net income is associated with an indifference curve, say $U_{c,1}$, that is lower than indifference curve U_1, but the increase in net income required to bring utility up to U_0 is a minimum. This is shown in Figure 3 by the length DC.

[7] A convex range occurs if the marginal tax rate falls as hours of work increase. This may happen when entitlement to a means-tested benefit is exhausted.

Let the virtual wage associated with the hours level h_c, at point C on indifference curve U_0 in Figure 3 be denoted w_c. Similarly, let the virtual wage for hours h_c on $U_{c,1}$ at point D be $w_{c,1}$. The virtual incomes associated with these two points are therefore expressed as $\mu(w_c, U_0)$ and $\mu(w_{c,1}, U_{c,1})$. Hence the compensating variation is:

$$CV = \{\mu(w_c, U_0) + w_c h_c\} - \{\mu(w_{c,1}, U_{c,1}) + w_{c,1} h_c\} \tag{10}$$

This can be rewritten as:

$$\begin{aligned} CV &= \{\mu(w_1, U_0) - \mu_1\} \\ &\quad + \{\mu(w_c, U_0) - \mu(w_1, U_0)\} \\ &\quad + \{w_c - w_1\} h_c \\ &\quad + \{w_1 - w_{c,1}\} h_c \\ &\quad + \{\mu_1 - \mu(w_{c,1}, U_{c,1})\} \end{aligned} \tag{11}$$

The first three terms in curly brackets in (11) correspond to those in equation (8). It is therefore clear that the previous subsection considered a special case for which $h_c = h_1$. Where $h_c < h_1$ the compensating variation is lower, as a result of the final two lines in (11).[8]

This can be rearranged into full income components as follows:

$$\begin{aligned} CV &= \{M(w_1, U_0) - M_0\} + \{M_0 - M_1\} \\ &\quad + \{M(w_c, U_0) - M(w_1, U_0)\} \\ &\quad + \{w_1 - w_c\}(T - h_c) \\ &\quad + \{w_{c,1} - w_1\}(T - h_c) \\ &\quad + \{M_1 - M(w_{c,1}, U_{c,1})\} \end{aligned} \tag{12}$$

which can be compared with (9).

The same kind of argument can be applied to the equivalent variation, where it is required to maximise the change in income necessary to reach U_1

[8] If $U_0 > U_1$, then $h_c \leq h_1$.

from the initial budget constraint under tax system 0. It can be shown that the general result for the equivalent variation is as follows:

$$\begin{aligned} EV &= \{M_1 - M(w_0, U_1)\} + \{M_0 - M_1\} \\ &\quad + \{M(w_0, U_1) - M(w_e, U_1)\} \\ &\quad + \{w_e - w_0\}(T - h_e) \\ &\quad + \{w_0 - w_{e,0}\}(T - h_e) \\ &\quad + \{M(w_{e,0}, U_{e,0}) - M_0\} \end{aligned} \tag{13}$$

where $h_e, w_e, w_{e,0}$ and $U_{e,0}$ are the equivalents of $h_c, w_c, w_{c,1}$ and $U_{c,1}$.

The calculation of h_c and h_e requires a systematic search over the corners and linear sections of the precise budget constraint (ignoring corners and linear segments associated with non-convex ranges of the budget set). The distances from the possible corners of the new budget constraint (at the hours levels where the changes in marginal tax rates apply) to the indifference curve U_0 are compared. Furthermore, possible tangency solutions for linear sections of the new budget constraint, shifted vertically upwards to touch U_0, are investigated. This is done for each segment by comparing the range of marginal rates of substitition, over the hours interval, with the net wage for the segment. If a tangency is found to exist, the precise hours and net income are obtained by simultaneously solving the two relevant equations.[9] Depending on the form of the utility function, an iterative numerical procedure may be needed to solve these equations. In the case of the compensating variation, the compensation needed is obtained by minimising the distance in selecting h_c. For the equivalent variation, the corresponding vertical distance is maximised in obtaining h_e.

[9] Determine the tangencies to U_0 by solving c and h from $\left.\frac{dc}{dh}\right|_U = w_1$ and $U(c,h) = U_0$, for each linear segment of the budget constrant for which w_1 lies between the virtual wage rates at the surrounding corners of the segment.

3 Discrete Hours Choices

Instead of allowing the hours worked to be varied continuously, this section examines a discrete hours model in which individuals are restricted to a limited number of hours levels, $h_1, ..., h_H$. The parameters of the utility function are available along with the net incomes at each of the specified hours points. Evaluation of the optimal number of hours is therefore easily carried out by calculating utilities at a relatively small number of points.[10] Each of these points is treated as a corner solution.

The discussion of piecewise-linear constraints where there is a corner solution, as in the previous section, suggests that in the case of discrete hours it is again no longer appropriate to use the standard formulae for the welfare changes. The degree of substitution between hours levels is substantially restricted, with the implication that compensation has to be greater than otherwise.

3.1 The Framework of Analysis

Consider Figure 4, where it is assumed that there are just four discrete hours levels available. Here it is useful to introduce a new notation system, in view of the use of $h_1, ..., h_H$ to refer to fixed discrete hours levels and the possible need to consider more than two indifference curves. In addition to the subscript denoting the tax system, each indifference curve is given a superscript which refers to the hours level; that is U_j^k is the utility obtained from combination of the discrete hours level h_k and the associated consumption determined by tax and transfer system j.

A similar convention is used when referring to virtual incomes and virtual wages, defined in terms of the tangent to a given indifference curve at a specified hours level. Supercripts indicate the hours index which defines the

[10] This contrasts with the continuous hours case where an efficient search algorithm, such as the one described in Creedy and Duncan (2001), may be adopted.

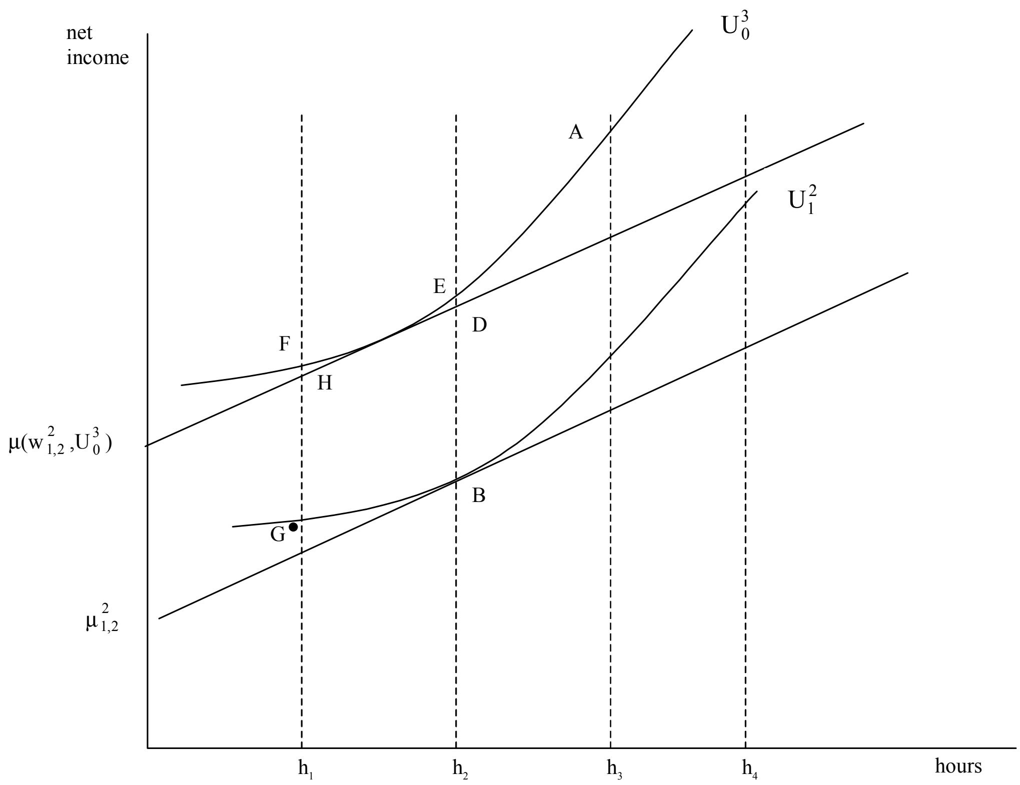

Figure 4: Discrete Hours

utility level, while subscripts refer to the discrete hours level to which the virtual values relate. Hence the virtual wage $w_{0,j}^k$ is the slope of indifference curve U_0^k at the discrete hours point h_j. Similarly, $\mu_{0,j}^k$ is the corresponding virtual income, that is, the intercept on the net income axis of the tangent to the indifference curve U_0^k at the discrete hours level h_j.

3.2 Exact Welfare Changes

The original optimal position in Figure 4 is at point A on indifference curve U_0^3, corresponding to h_3 hours of work. Suppose a tax reform causes the optimal position to shift to point B on indifference curve U_1^2, involving h_2 hours of work. Hence the virtual budget constraints associated with A and B are defined by the pairs $(\mu_{0,3}^3, w_{0,3}^3)$ and $(\mu_{1,2}^2, w_{1,2}^2)$ respectively.[11] Given the limited hours choices available, calculation of the compensating variation using the standard approach understates the true amount needed to restore the individual to U_0^3. The exact approach used when discussing continuous hours variations can be directly applied here. The compensation is the difference between the net incomes at points E and B.

Using the convention described above, $w_{0,2}^3$ denotes the virtual wage corresponding to hours level h_2 at point E on U_0^3 and $\mu_{0,2}^3$ represents the associated virtual income.[12] Compensation required, the length EB, is given by:

$$CV = \left\{\mu_{0,2}^3 + w_{0,2}^3 h_2\right\} - \left\{\mu_{1,2}^2 + w_{1,2}^2 h_2\right\} \tag{14}$$

or equivalently:

$$CV = \left\{\mu_{0,2}^3 - \mu_{1,2}^2\right\} + \left(w_{0,2}^3 - w_{1,2}^2\right) h_2 \tag{15}$$

In terms of associated full incomes, this can be written as:

$$CV = \left(\mu_{0,2}^3 + w_{0,2}^3 T\right) - \left(\mu_{1,2}^2 + w_{1,2}^2 T\right) + \left(w_{1,2}^2 - w_{0,2}^3\right)(T - h_2) \tag{16}$$

[11] For each discrete hours point, h_j, c_j^0 and c_j^1 can be determined, after which $U\left(c_j^0, h_j\right)$ can be calculated. Then $w_{0,i}^i$ is the virtual wage in the optimal point U_0^i and $\mu_{0,i}^i = c_i^0 - w_{0,i}^i h_i$.

[12] Determine net income $c_{0,2}^3$ necessary to reach U_0^3 in h_2 by solving for c in $U(c, h_2) = U_0^3$ and then use $\mu_{0,2}^3 = c_{0,2}^3 - w_{0,2}^3 h_2$.

This assumes that the individual remains at discrete hours level h_2. However, the possibility must be considered that the individual could work h_1 hours and reach indifference curve U_0^3 with a smaller increase to net income than the value given in (14). If net income at h_1 is at point G (which is above the virtual budget constraint associated with B), it is possible that the distance between G and F is smaller than that between B and E. Even if G is slightly below the virtual budget line through B, it is still possible for the compensating variation to be lower than if hours were fixed at h_2, depending on the distance FH compared with ED.

Point G is the combination of net income on the actual budget constraint under the post-reform tax system and the hours level h_1, so the indifference curve through this point is labelled U_1^1. If the individual is at G, the compensation required to reach U_0^3, the length GF, is given by:

$$CV = \left\{\mu_{0,1}^3 + w_{0,1}^3 h_1\right\} - \left\{\mu_{1,1}^1 + w_{1,1}^1 h_1\right\} \tag{17}$$

or equivalently:

$$CV = \left\{\mu_{0,1}^3 - \mu_{1,1}^1\right\} + \left(w_{0,1}^3 - w_{1,1}^1\right) h_1 \tag{18}$$

The appropriate compensation is therefore the minimum of this type of difference, over all discrete hours points. The search for the appropriate hours level which produces a minimum compensation is of course simpler in the discrete hours context, where all possible hours levels are specified *a priori*, compared with the continous hours framework (where an alternative tangency may apply). In the examples given below, the appropriate hours levels for compensating and equivalent variations (from which the welfare change is the minimum difference between net income and the relevant indifference curve) are referred to as h_c and h_e, by analogy with their earlier use in the continous hours context.

If a utility function is used for which the labour supply and expenditure functions cannot be derived explicitly, it is clearly not possible to obtain an analytical expression for the virtual income $\mu\left(w_{1,2}^{2},U_{0}^{3}\right)$, shown in Figure 4, or the associated expenditure function expressed in terms of full income. But this does not matter as it is not actually required in the discrete hours context.

3.3 Computational Requirements

The use of a discrete hours framework has substantial advantages from the point of view of econometric estimation of preferences. In addition to avoiding the need to deal with the endogeneity of net wage rates (compared with estimation of a continuous labour supply function where hours worked are expressed as a continuous function of net wage and virtual income), and the need to compute net incomes over the complete hours range for each individual, it is not necessary to solve the first-order conditions for maximising utility subject to the (linear) budget constraint. Hence an explicit form for the labour supply function is not needed. This means that a wider range of direct utility functions can be used.[13]

The same advantage applies to the calculation of welfare changes. The procedure outlined above requires only the calculation of virtual wage rates, virtual incomes and the consumption (net income) level corresponding to a specified hours level and indifference curve (or utility level). These are obtained as follows.

First, the virtual wage, w, at any point (defined by a combination of c and h) is given simply by the slope of the indifference curve at that point. Hence:

$$w=\left.\frac{dc}{dh}\right|_{U(c,h)} \tag{19}$$

and this can usually be obtained with little difficulty. Second, the corre-

[13] In the continuous case, even if estimation is based on an explicit labour supply function (expressed in terms of μ and w), it is necessary for welfare measurement to integrate from the labour supply to the expenditure function. This integration may need to be carried out numerically.

sponding virtual income, $\mu(w, U)$, can be calculated using the equation of the virtual linear budget constraint. Using the combination of w, c and h in (19), the corresponding virtual income is:

$$\mu(w, U) = c - hw \tag{20}$$

Third, it is necessary to compute the value of net income (consumption), c, corresponding to a specified hours level, h, along an indifference curve with known utility level, U (computed from a known different optimal combination of consumption and hours).

4 The Quadratic Direct Utility Function

The quadratic utility function has been used in empirical analyses of labour supply in the context of discrete hours models. Examples include Keane and Moffitt (1998), Duncan and Weeks (1997, 1998), and Duncan and Giles (1998). The following subsections describe the main properties of this function and present examples in both continuous and discrete hours contexts.

4.1 The Main Properties

The quadratic direct utility function is:

$$U = \alpha c^2 + \beta h^2 + \gamma ch + \delta c + \varepsilon h \tag{21}$$

and:

$$\left.\frac{dc}{dh}\right|_{U(c,h)} = -\frac{2\beta h + \gamma c + \varepsilon}{2\alpha c + \gamma h + \delta} \tag{22}$$

It is required to compute the value of net income (consumption), c, corresponding to a specified hours level, h, along an indifference curve with known utility level, U_0^i (computed from a known different combination of consumption and hours resulting in the optimal utility level). This is obtained as the appropriate root of the quadratic:

$$ac^2 + bc + d = 0 \tag{23}$$

with:

$$
\begin{aligned}
a &= \alpha \\
b &= \gamma h + \delta \\
d &= \beta h^2 + \varepsilon h - U
\end{aligned}
\tag{24}
$$

In practice the appropriate root is obvious.

Explicit analytical expressions can be obtained for the labour supply and expenditure functions in the case of the quadratic direct utility function. It is shown in Creedy (2000b) that the expenditure function, in terms of virtual income, is given by the smallest root of the quadratic:

$$
\mu^2 q_1 + \mu q_2 + q_3 = 0 \tag{25}
$$

where:

$$
\begin{aligned}
q_1 &= \alpha - \frac{(2\alpha w + \gamma)^2}{4(\alpha w^2 + \gamma w + \beta)} \\
q_2 &= \delta - \frac{(2\alpha w + \gamma)(\delta w + \varepsilon)}{2(\alpha w^2 + \gamma w + \beta)} \\
q_3 &= -\left(U + \frac{(\delta w + \varepsilon)^2}{4(\alpha w^2 + \gamma w + \beta)}\right)
\end{aligned}
\tag{26}
$$

This expenditure function facilitates comparison with the standard approach, in the continuous hours case.

4.2 Numerical Examples

In order to demonstrate the above procedures, this subsection presents a number of numerical examples based on simplified tax structures. Three schedules are shown in Table 1. There are five effective marginal tax rates, t_i, and corresponding income thresholds, y_i, above which these rates apply; these thresholds are the same for all structures. For example, structure 1 has a range where the marginal effective tax rate is zero, but after the threshold of $y_2 = 30$, the rate becomes 0.5; this can be regarded as a taper rate applying to a means-tested transfer payment. At an earnings threshold of $y_3 = 70$, the effective marginal rate drops to 0.3, when entitlement to the means-tested

Table 1: Hypothetical Tax Structures

		Structure 1		Stucture 2		Structure 3	
Segment	y_i	t_i	μ_i	t_i	μ_i	t_i	μ_i
1	0	0.0	100	0.0	**75**	0.0	100
2	30	0.5	115	0.5	**90**	0.5	115
3	70	0.3	101	0.3	**76**	0.3	101
4	200	0.4	121	0.4	**96**	**0.5**	**141**
5	350	0.5	156	0.5	**131**	**0.6**	**196**

benefit is exhausted. This produces a range where the budget set is non-convex. Beyond this range, the marginal tax rate increases. The transfer payment received by non-workers, μ_1, is set equal to 100. Given these values, the remaining virtual incomes, μ_i, can be calculated, and are shown in Table 1. In structure 2, non-wage income for non-workers, μ_1, is reduced from 100 to 75, while all income thresholds and marginal tax rates remain constant. The new virtual incomes are shown in bold in the table. The third tax structure has higher marginal rates in the 4th and 5th segments; the new values are also shown in bold. The following examples relate to two policy changes, from structure 1 to 2, and from 1 to 3.

Consider an individual with a quadratic direct utility function of the form:

$$U(h,c) = -0.007c^2 - 0.005h^2 + 5.5c - 2.5h \tag{27}$$

Continuous and discrete hours frameworks are examined in turn.

4.2.1 Continuous Hours

Suppose, first, that the individual has a gross wage of $w_g = 8$. With the tax structure of Table 1, the hours levels at which the effective marginal tax rates change are 0, 3.75, 8.75, 25.00 and 43.75. It is found that the gross wage of 8 gives rise to an optimal labour supply given by a tangency solution on section 5 of the constraint, corresponding to $h = 46$. The reduction in μ_1 to 75 means that the optimal position involves a tangency position on section 5 of the new constraint, where h rises to 51.98. The fact that the same section is relevant means that the net wage is unchanged, so there is only an income effect.[14] Both compensating and equivalent variations are equal to the change in virtual and full income, which is 25. Hence, for this particular case, the standard method of calculating welfare changes gives exactly the same results as in the general case discussed in section 2 above.

The second policy change involves an increase in labour supply to 47.05 hours, which is also on section 5 of the budget constraint. The standard method of computing the welfare change gives $CV = 15.549$, as shown in the second column of Table 2. However, the associated hours would take the individual beyond the 5th segment and into the 4th segment of the new budget constraint. Hence, the method proposed above gives values associated with the corner, at $h_c = 43.75$ hours. The various components of the welfare change are given in Table 2, where full incomes are computed using $T = 80$. This table shows that the exact CV is 15.687. However, in considering the equivalent variation, an hours level of $h_e = 50.60$ is calculated, which is still on the 5th linear segment. Hence, the standard method gives the correct value, as seen by the fact that the other components in Table 2 are zero.

Suppose next that the gross wage rate is $w_g = 5$. This implies that, for the tax system in Table 1, the hours levels at which the five marginal rates start to apply are 0, 6, 14, 40, and 70. The optimal position is found to be a tangency solution along section 4 of the budget constraint, corresponding to $h = 65.57$. For the first policy change described above, where there is only a change in μ_1, the new optimal position involves a shift from the tangency position on segment 4 to a corner solution at the start of segment 5, with $h = 70$. The shift to a corner solution means that both the endogenous virtual wage and income change. The standard method of computing welfare changes gives $CV = 22.777$ and $EV = 25.685$, as shown in Table 2. However, in this case

[14] Thus $h_c = h_0$ and $h_e = h_1$.

Table 2: Welfare Changes for Continuous Hours

	Policy 2 $w_g = 8$	Policy 1 $w_g = 5$
Compensating Variation		
$\{M(w_1, U_0) - M_0\} + \{M_0 - M_1\}$	15.549	22.777
$\{M(w_c, U_0) - M(w_1, U_0)\}$	8.482	5.907
$\{w_1 - w_c\}(T - h_c)$	−8.344	−5.315
$\{w_{c,1} - w_1\}(T - h_c)$	−17.037	5.315
$\{M_1 - M(w_{c,1}, U_{c,1})\}$	17.037	−3.684
CV	15.687	25
Equivalent Variation		
$\{M_1 - M(w_0, U_1)\} + \{M_0 - M_1\}$	19.207	25.685
$\{M(w_0, U_1) - M(w_e, U_1)\}$	0	3.000
$\{w_0 - w_{e,1}\}(T - h_e)$	0	−6.9525
$\{w_e - w_0\}(T - h_e)$	0	−3.684
$\{M(w_{e,1}, U_{e,0}) - M_0\}$	0	6.952
EV	19.207	25

the assumptions on which the standard welfare measures are based do not hold; these measures are shown in the table. Using (12), the compensating variation, allowing for the precise nonlinearity of the budget constraint, is 25, equal to the reduction in μ_1. In this case, $h_c = h_0$ and $h_e = h_1$. Similarly, the equivalent variation is also equal to 25.

For the second policy change and the individual with a gross wage of $w_g = 5$, labour supply falls to 64.77, but this is also a tangency solution along the fourth segment. The fall in full income is 20, while, following the standard approach, the welfare gains associated with the fall in the price of leisure are 8.356 and 6.248, for compensating and equivalent variations respectively. Thus $CV = 11.64$ and $EV = 13.75$. For this case, $h_c = 60.58$ and $h_e = 69.82$. Given that all the relevant hours positions involve tangency solutions along the same segment, the standard measures are exactly the same as those produced by the general approach.

Table 3: Welfare Changes for Wage Rate of 8

hours	standard				exact					
steps	CV	EV	h_c	h_e	CV	EV	h_c	h_e	h_0	h_1
First tax change: reduction in μ_1										
5	22.32	23.53	45	50	25	25	45	50	45	50
10	21.85	25.00	40	50	25	25	50	50	50	50
20	-	15.27	40	40	25	25	40	60	40	60
Second tax change:										
5	14.37	18.09	40	50	16.00	19.56	45	50	45	45
10	19.89	20.00	40	50	17.21	20.00	40	50	50	50
20	12.00	12.00	40	40	12.00	12.00	40	40	40	40

4.2.2 Discrete Hours

Table 3 shows alternative welfare measures obtained for the tax policy changes considered above, where the gross wage is 8. The table gives results using discrete hours that are limited to intervals of 5, 10 and 20. For the first policy change, and each set of discrete hours intervals, the exact measures coincide with the continuous hours context, giving an income effect only, whereby equivalent and compensating variations are 25. In view of the fact that the restriction to discrete hours causes a deviation of the hours level from the level that would be optimal under continuous hours models, the standard approach can lead to values that deviate substantially when the discrete interval is large. Indeed, the CV when hours intervals of 20 are imposed, is meaningless because it involves a range of the utility function where the standard conditions do not apply (for example, the indifference curves slope in the 'wrong' direction).

Table 4 shows results for a gross wage rate of 5 and the two policies. Again the exact approach gives results for the first policy that are significantly different from those obtained using the standard approach. For the second policy, involving changes in two of the marginal tax rates, the alternative approaches are much closer together, because the values of h_c and h_e are the

Table 4: Welfare Measures for Wage Rate of 5

hours	standard				exact					
steps	CV	EV	h_c	h_e	CV	EV	h_c	h_e	h_0	h_1
First tax change: reduction in μ_1										
5	22.95	24.81	60	75	25	25	65	70	65	70
10	21.92	28.95	60	80	25	25	70	70	70	70
20	46.21	22.03	60	80	25	25	60	60	60	80
Second tax change:										
5	11.73	13.39	60	70	11.64	13.75	60	70	65	65
10	10.61	14.79	60	70	10.61	14.79	60	70	70	60
20	10.00	10.00	60	60	10.00	10.00	60	60	60	60

same for both the standard and exact methods.

For both gross wage rates and the second policy, the case where the discrete hours interval is 20 hours gives rise to equal values for both the EV and the CV for standard and exact methods. Here, $h_0 = h_1 = h_c = h_e$ for each measure. This is not surprising in view of the fact that there are so few hours positions available.

5 The Quadratic Translog Utility Function

A number of econometric studies of labour supply behaviour, using a discrete set of hours, have estimated the translog form of the quadratic utility function in which utility is expressed as a function of the square of the logarithms of consumption, and so on; see van Soest (1995), Callan and Van Soest (1996), Van Soest and Das (2000) and Kalb (1999, 2000).[15] However, these studies have not attempted to measure the welfare changes arising from reforms to the tax and transfer system.

[15]The logarithmic transformation can be awkward where fixed costs of working are included, since circumstances exist where consumption can be negative. The integrability condition is not automatically satisfied for this model, or for the simpler quadratic utility function.

5.1 The Main Properties

As before, let c denote consumption (net income) and h the number of hours worked. The direct utility function $U = U(c, h)$ must in this case be written in terms of leisure, $T - h$, in order to avoid taking the logarithm of zero for non-workers. Hence:

$$U = \alpha (\log c)^2 + \beta (\log (T-h))^2 + \gamma (\log c)(\log (T-h)) + \delta \log c + \varepsilon \log (T-h) \tag{28}$$

The virtual wage, w, at any point (defined by a combination of c and h) is given simply by the slope of the indifference curve at that point. Hence, from (19):

$$w = \left.\frac{dc}{dh}\right|_{U(c,h)} = \left(\frac{c}{T-h}\right)\left(\frac{2\beta \log (T-h) + \gamma \log c + \varepsilon}{2\alpha \log c + \gamma \log (T-h) + \delta}\right) \tag{29}$$

Convenient analytical expressions cannot be obtained for optimal c and h, given the budget constraint $c = wh + \mu$, so that the indirect utility and expenditure functions cannot be derived.[16] Comparisons with the standard approach in the continuous hours case is therefore complicated by the need to solve the simultaneous equations $c = wh + \mu$ and $w = \left.\frac{dc}{dh}\right|_{U(c,h)}$ numerically.

Using the combination of w, c and h used in (29), the corresponding virtual income is, as mentioned earlier, $\mu(w, U) = c - hw$.

To compute the value of net income (consumption), c, corresponding to a specified hours level, h, along an indifference curve with given utility level, U, let $x = \log c$. The required value of x is given by obtaining the appropriate root of the quadratic:

$$ax^2 + bx + d = 0 \tag{30}$$

with:

$$\begin{aligned} a &= \alpha \\ b &= \gamma \log (T-h) + \delta \\ d &= \beta (\log (T-h))^2 + \varepsilon \log (T-h) - U \end{aligned} \tag{31}$$

[16] Stern (1986, p.184) devotes only one line to this case, saying that it ‘gives intractable supply functions’.

whence the required consumption is given by $c = \exp x$. It is found that the largest root is without ambiguity the appropriate one to take.

5.2 Numerical Examples

Consider an individual with a translog quadratic utility function of the form:

$$U = 0.427\,(\log c)^2 + 0.828\,(\log(T-h))^2 - 0.340\log c - 3.183\log(T-h) \quad (32)$$

In this case T is set at 100 hours.[17]

5.2.1 Continuous Hours

Using the same tax structures as for the previous examples, results for the context of continuous hours are shown in Table 5 for both the standard method and the exact method, for each wage rate. The computation of welfare changes for the continuous case requires, as mentioned earlier, the use of numerical solution procedures in view of the fact that the labour supply and expenditure functions cannot be derived analytically. The optimal hours under both the initial tax structure and structure 2 for $w_g = 5$ is at the corner solution where $h = 40$, at the end of segment three and the start of segment four. Hence the exact method involves only an income effect, with $h_c = h_e = 40$. This differs from the value obtained using the standard approach. The second policy change for this wage rate has no effect since the new marginal rates are irrelevant.

For $w_g = 8$, the first two tax structures both give rise to an optimal solution at the corner given by the start of segment five, where $h_0 = h_1 = 43.75$. The standard and exact solutions therefore differ, since the latter involves only an income effect, again equal to the change in μ_1 of 25. The second policy change for $w_g = 8$ involves a movement from the corner solution to a tangency solution along segment four, with $h_1 = 40.25$. Calculation of

[17] This avoids any chance of having $T - h < 0$.

Table 5: Continuous Hours: Quadratic Translog Utility

	$w_g = 5$				$w_g = 8$			
	Standard		Exact		Standard		Exact	
	CV	EV	CV	EV	CV	EV	CV	EV
Policy 1	24.39	25.60	25	25	24.57	25.43	25	25
Policy 2	0	0	0	0	13.82	14.64	13.82	14.50

the compensating variation in the standard case involves an hours level of 38.37, which is on the same segment: hence the standard and exact methods give the same result, with $h_c = 38.37$. However, for the equivalent variation, the relevant hours level is beyond that at which the fifth segment begins, so the two methods give different welfare change values. Here the exact method is associated with $h_e = 43.75$, that is at the kink.

5.2.2 Discrete Hours

Table 6 gives results of using the different methods in the discrete hours context, using three alternative discrete hours intervals, for a gross wage rate of 8. Here there are several cases where the welfare changes involve only an income effect, whereby all measures are equal. This is true even where, as in the second policy change, a marginal tax rate changes; the discrete hours context means that, with fewer hours levels available, the associated hours levels are all equal.

Table 7 gives results for the discrete hours context with a wage of $w_g = 5$. Here results are shown only for the first policy change, since all welfare changes for the second policy change are (as in the continuous hours context) zero. With the first policy change and hours intervals of 10 and 20, all welfare changes are again equal because all the relevant hours levels are the same, at 40 hours.

Table 6: Welfare Measures for Wage Rate of 8

hours	standard				exact					
steps	CV	EV	h_c	h_e	CV	EV	h_c	h_e	h_0	h_1
First tax change: reduction in μ_1										
5	24.72	25.23	40	50	25	25	45	45	45	45
10	27.54	24.64	50	40	25	25	40	50	40	50
20	25	25	40	40	25	25	40	40	40	40
Second tax change:										
5	13.32	14.07	40	45	13.32	14.07	40	45	45	40
10	12.00	12.00	40	40	12.00	12.00	40	40	40	40
20	12.00	12.00	40	40	12.00	12.00	40	40	40	40

Table 7: Welfare Measures for Wage Rate of 5

hours	standard				exact					
steps	CV	EV	h_c	h_e	CV	EV	h_c	h_e	h_0	h_1
First tax change: reduction in μ_1										
5	24.40	25.59	35	45	25	25	40	40	40	40
10	25.00	25.00	40	40	25	25	40	40	40	40
20	25.00	25.00	40	40	25	25	40	40	40	40

6 Conclusions

This paper has examined the calculation of compensating and equivalent variations in the context of labour supply, where highly nonlinear (typically piecewise-linear) tax functions are normal. It was shown that the standard method of computing these changes may not give appropriate values, if they involve hours levels for which the linearised virtual budget constraint indicates a different net income from the exact nonlinear budget constraint, or when corner solutions are involved. Methods of calculating exact welfare changes, allowing for the full detail of the budget constraint, were proposed in the context of both continuous and discrete hours models. The need to consider discrete hours models is important because these are being more widely adopted as a result of their substantial advantages in preference estimation.

It was shown that exact welfare changes require the use of a search procedure in which comparisons are made, for all appropriate corner and tangency solutions, of the vertical distance between net income on the relevant budget constraint and that on the relevant indifference curve. In the continuous hours context, this method may require the use of numerical solution procedures to examine the tangency positions, depending on the choice of utility function. Numerical examples were given using the basic and translog versions of the quadratic direct utility function, for continuous and discrete hours frameworks. The use of the standard method in the discrete case may result in unrealistic outcomes, particularly when the hours intervals become large. In such cases the exact approach still provides sensible results.

References

[1] Blomquist, N.S. (1983) The effect of income taxation on the labour supply of married men in Sweden. *Journal of Public Economics*, 22, pp. 169-197.

[2] Blundell, R., Preston, I. and Walker, I. (eds) (1994) *The Measurement of Household Welfare*. Cambridge: Cambridge University Press.

[3] Callan, T. and Van Soest, A. (1996) Family labour supply and taxes in Ireland. (Tilburg University).

[4] Creedy, J. (2000a) Measuring welfare changes and the excess burden of taxation. *Bulletin of Economic Research,* 52, pp. 1-47.

[5] Creedy, J. (2000b) Quadratic utility, labour supply and the welfare effects of tax changes. *University of Melbourne Department of Economics Research Paper.*

[6] Creedy, J. and Duncan, A. S. (2001) Behavioural microsimulation with labour supply responses. *Journal of Economic Surveys*, (forthcoming).

[7] Duncan, A. S. and Giles, C. (1998) The labour market inpact of the working families tax credit in the UK. (Paper presented to International Institute for Public Finance conference, Cordoba, Argentina).

[8] Duncan, A. S. and Weeks, M. (1997) Behavioural tax microsimulation with finite hours choices. *European Economic Review*, 41, pp. 619-626.

[9] Duncan, A. S. and Weeks, M. (1998) Simulating transitions using discrete choice models. *Proceedings of the American Statistical Association*, 106, pp. 151-156.

[10] Hausman, J.A. (1981) Exact consumer's surplus and deadweight loss. *American Economic Review*, 71, pp. 662-676.

[11] Hausman, J.A. (1985) Taxes and labor Supply. In *Handbook of Labor Economics, Vol. 1.* (ed. by A.J. Auerbach and M. Feldstein), pp.213-264. New York: North-Holland.

[12] Johansson, P-O. (1987) *The Economic Theory and Measurement of Environmental Benefits*. Cambridge: Cambridge University Press.

[13] Kalb, G. (1999) Labour supply and welfare participation in Australian two-adult households; comparing 1986/1987 with 1994/1995. *Centre of Policy Studies/Impact Project working paper series BP*-3497.

[14] Kalb, G. (2000) Labour supply and welfare participation in Australian two-adult households: accounting for involuntary unemployment and the cost of part-time work. *Centre of Policy Studies/Impact Project working paper series BP*-3597.

[15] Keane, M., and Moffitt, R. (1998) A structural model of multiple welfare program participation and labour supply. *International Economic Review,* 39, pp. 553-590.

[16] Stern, N. (1986) On the specification of labour supply functions. In *Unemployment, Search and Labour Supply* (ed. by R. Blundell and I. Walker), pp. 143-189). Cambridge: Cambridge University Press.

[17] Van Soest, A. (1995) Discrete choice models of family labour supply. *Journal of Human Resources,* 30, pp. 63-88.

[18] Van Soest, A. and Das, M. (2000) Family labour supply and proposed tax reforms in the Netherlands. In *Taxes, Transfers and Labour Market Responses: What Can Microsimulation Tell Us?* (ed. by T. Callan), pp. 59-80. Dublin: Economic and Social Research Institute

[19] Vlasblom, J.D., De Gijsel, P. and Siegers, J. (2001) Taxes, female labour supply and household income: differences between the Netherlands and the Federal Republic of Germany. *Applied Economics*, 33, pp. 735-744.