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Abstract

We propose a novel identification-robust test for the null hypothesis that an estimated new-Keynesian model has a reduced form consistent with the unique stable solution against the alternative of sunspot-driven multiple equilibria. Our strategy is designed to handle identification failures as well as the misspecification of the relevant propagation mechanisms. We invert a likelihood ratio test for the cross-equation restrictions (CER) that the new-Keynesian system places on its reduced form solution under determinacy. If the CER are not rejected, sunspot-driven expectations can be ruled out from the model equilibrium and we accept the structural model. Otherwise, we move to a second-step and invert an Anderson and Rubin-type test for the orthogonality restrictions (OR) implied by the system of Euler equations. The hypothesis of indeterminacy and the structural model are accepted if the OR are not rejected. We investigate the finite sample performance of the suggested identification-robust two-steps testing strategy by some Monte Carlo experiments and then apply it to a new-Keynesian AD/AS model estimated with actual U.S. data. In spite of some evidence of weak identification as for the 'Great Moderation' period, our results offer formal support to the hypothesis of a switch from indeterminacy to a scenario consistent with uniqueness which occurred in the late 1970s. Our identification-robust full-information confidence set for the structural parameters computed on the 'Great Moderation' regime turns out to be more precise than the intervals previously reported in the literature through 'limited-information' methods.

JEL classification: C31, C22, E31, E52

Keywords: Confidence set, determinacy, identification failures, indeterminacy, misspecification, new-Keynesian business cycle model, VAR system

1 Introduction

The U.S. inflation and output growth processes have experienced dramatic breaks in the post-WWII. In particular, a marked reduction of the U.S. macroeconomic volatilities has been documented by Stock and Watson (2002), who coined the popular term 'Great Moderation' to indicate this stylized fact. A possible explanation for such phenomenon hinges upon the hypothesis of the switch to an aggressive monetary policy conduct occurred with the appointment of Paul Volcker as Chairman of the Federal Reserve at the end of the 1970s. With his appointment, the argument goes, the Fed moved from a weakly aggressive reaction to inflation to a much stronger one. Such a switch anchored private sector's inflation expectations, therefore leading the U.S. economy to move from an indeterminate equilibrium to determinacy. This story, popularized by Clarida *et al.* (2000), has subsequently been supported by Lubik and Schorfheide (2004), Boivin and Giannoni (2006), Benati and Surico (2009), Mavroeidis (2010), and Inoue and Rossi (2011*a*).

The above mentioned contributions implicitly assume the new-Keynesian model one works with to be correctly specified and, with the remarkable exception of Mavroeidis (2010), to feature identifiable parameters. As concerns the first issue, albeit new-Keynesian models can display several types of misspecification (An and Schorfheide, 2007), the omission of propagation mechanisms from the structural equations is a major concern in the empirical assessment of determinacy/indeterminacy. As discussed by Lubik and Schorfheide (2004) and Fanelli (2012), indeterminacy generally entails a richer correlation structure of the data. Therefore, the risk run by an econometrician is to confound a determinate case in which relevant propagation mechanisms are not embedded by the structural model at hand with the indeterminate scenario. In conducting their Bayesian analysis, Lubik and Schorfheide (2004) tackle this issue by analyzing versions of a small-scale new-Keynesian model featuring different dynamic structures, while Fanelli (2012) proposes a frequentist test of determinacy/indeterminacy that explicitly controls for the omission of propagation mechanisms from the specified system of structural Euler equations.

As concerns the identifiability of the structural parameters, aside from Mavroeidis (2010), who adopts a single-equation 'limited-information' approach, all existing empirical contributions in which the determinacy/indeterminacy issue of U.S. monetary policy is investigated assume that the structural parameters are identifiable. In general, both finite sample and asymptotic distributions for estimators and tests can be strongly affected if identification conditions are not satisfied, see e.g. Sargan (1983), Phillips (1989), Staiger and Stock (1997) and Stock and Wright (2000). Many authors have recently argued that estimated new-Keynesian systems like or similar to the one considered in this paper can be affected by 'weak identification' issues. Identification problems in a system of variables featuring highly nonlinear restrictions may involve the rank condition of the information matrix or suitable transformation of moments (Iskrev, 2008, 2010; Komunjer and Ng, 2011), or the relationship between the structural parameters and the sample objective function, which may display 'small' curvature in certain regions of the parameter space, see e.g. Canova and Sala (2009). The former concept of identification is also referred to as 'population identification', as opposed to the latter, often termed 'sample identification', because it is specific to a particular dataset and sample size (for proponents of this terminology, see Canova and Sala, 2009). Our paper is concerned with this second phenomenon, which we characterize as the situation in which the criterion used to estimate the structural parameters and test hypotheses on these parameters exhibit 'little curvature' in all or some directions of the parameter space, with the consequence of being nearly uninformative about these parameters. Weak identification of all or part of the estimated parameters can affect negatively the finite sample performances of the testing procedures commonly used by 'frequentist' practitioners. Robust inference under possible identification failure has received increasing attention by the literature on dynamic stochastic general equilibrium (DSGE) models, see e.g. Canova and Sala (2009), Dufour *et al.* (2009, 2013), Kleibergen and Mavroeidis (2009), Mavroeidis (2005, 2010), Guerron-Quintana *et al.* (2013) and Andrews and Mikusheva (2014), among others.[1]

This paper's contribution is twofold. On the methodological side, we propose a novel identification-robust test for the null hypothesis that an estimated new-Keynesian model has a reduced form consistent with the unique stable solution, against the alternative of sunspot-driven multiple equilibria. The test (i) can be applied regardless of the strength of identification of the model's structural parameters, and (ii) controls for the case of 'dynamic misspecification', where by this term we mean the omission of relevant propagation mechanisms from the specified system of structural Euler equations. On the empirical side, we use the small scale new-Keynesian model discussed in Benati and Surico (2009) and apply the proposed method to post-WWII U.S. data to investigate indeterminacy issues in the conduct of monetary policy on our selected 'pre-Volcker' and 'Great Moderation' samples.

As regards the methodological contribution, the proposed testing strategy is based on two steps. In the first-step, we use an identification-robust 'full-

[1] Inoue and Rossi (2011*b*) and Andrews and Cheng (2012) tackle the issue from a more general perspective but their analysis can be adapted to the context of DSGE models.

information' method to test the cross-equation restrictions (CER) that the new-Keynesian model places on its unique stable reduced form solution under determinacy. This requires the (numerical) inversion of a likelihood-ratio test for the CER implied by the new-Keynesian model along the lines recently suggested by Guerron-Quintana *et al.* (2013) and Dufour *et al.* (2013). If the CER are not rejected, we can rule out the occurrence of sunspot-driven expectations and arbitrary nuisance parameters from the model's equilibrium. Importantly, in this case we cannot rule out the possibility of a Minimum State Variable (MSV) equilibrium (McCallum, 1983), i.e. a solution nested within the class of indeterminate equilibria observationally equivalent to the determinate reduced form, see Evans and Honkapohja (1986), Lubik and Schorfheide (2004) and Fanelli (2012). Notably, the non-rejection of the CER amounts to an implicit acceptance of the hypothesis of correct specification of the new-Keynesian system. If instead the CER are rejected, we move to a second-step to determine whether the outcome obtained in the first-step depends on the multiple equilibria hypothesis, or to the omission of relevant propagation mechanisms from the specified structural equations. We apply an identification-robust 'limited-information' method and invert a test for the orthogonality restrictions (OR) implied by the system of Euler equations under the rational expectations hypothesis (and the assumption of correct specification). In principle, if the new-Keynesian system is correctly specified, the OR are valid irrespective of whether the implied equilibrium is determinate or indeterminate. However, conditional on the result in first-step, the non-rejection of the OR is in our framework evidence of indeterminacy, while their rejection suggests that the specified structural equations do not capture the dynamic of the data sufficiently well. The test inverted in this second-step is an Anderson Rubin-type test (Anderson and Rubin, 1949) that can be implemented in the multivariate framework along the lines suggested by Dufour *et al.* (2009, 2013).[2]

The tests computed in both steps are based on asymptotically pivotal test statistics regardless of the strength of identification of the model's structural parameters. The overall testing strategy is asymptotically correctly sized. We investigate its finite sample performance by some Monte Carlo experiments, using the new-Keynesian model by Benati and Surico (2009) as data generating process.

As regards the empirical contribution, the application of our testing strategy on U.S. quarterly data using Benati and Surico's (2009) model as bench-

[2] Alternatively, one can apply the 'S-test' approach by Stock and Wright (2000) or the 'K-LM test' approach by Kleibergen (2005), which require the evaluation of the criterion function associated with the continuos-updating version of the generalized method of moments.

mark, leads us to the following findings. The identification-robust test for the CER computed in the first-step implies the rejection of the hypothesis of determinacy on the 'pre-Volcker' sample. Conditional on this first-step, our identification-robust test for the OR computed in the second-step does not reject the new-Keynesian framework at hand. Therefore, our results support the multiple equilibria scenario, which acknowledges a role for self-fulfilling expectations as a driver of the U.S. macroeconomic dynamics during the 1970s. Instead, when considering our 'Great moderation' sample, the identification-robust test for the CER computed in the first-step clearly supports the CER implied by the hypothesis of determinacy. While being unable to interpret this result as conclusive evidence of determinacy (recall the observational equivalence between the determinate and the indeterminate MSV solution), the case of sunspot shocks-driven expectations is clearly ruled out by the data. In line with Mavroeidis (2010), the 'limited-information' approach we implement in the second-step delivers wider projected confidence intervals for the policy parameters during the 'Great Moderation', as opposed to those computed for the 'Great Inflation' period. If taken in isolation, the projected confidence intervals of the policy parameters would be considered as uninformative as for the issue of determinacy. Differently, our 'full-information' inferential approach enables us to interpret such evidence as consistent with an economic system under determinacy, hence not affected by sunspot shocks. Therefore, our testing procedure is inherently more informative than a single-equation approach (even when the latter is designed to deal with weak identification), in that it allows the econometrician to go a step further in assessing (and, in this case, ruling out) the role of sunspot fluctuations as possible drivers of the U.S. economic dynamics.

The remained of this paper is organized as follows. Section 2 introduces the reference small scale new-Keynesian structural model, reports the time series representations of its reduced form solutions under determinacy (Sub-section 2.1) and indeterminacy (Sub-section 2.2), and analyzes the conditions under which observational equivalence occurs (Sub-section 2.3). Section 3 discusses how inference can be conducted under identification failure in a 'full-information' framework (Sub-section 3.1) and in a 'limited-information' (Sub-section 3.2) framework, and then combines these two approaches in a coherent testing strategy (Sub-section 3.3). Section 4 investigates the finite sample performance of the suggested testing strategy by some simulation experiments. Section 5 presents our empirical results obtained on U.S. quarterly data. Section 6 relates our work to the existing literature, and Section 7 contains some concluding remarks. Our Supplementary Material derives (i) the time series representations of the reduced form solutions associated with the new-Keynesian model, (ii) some asymptotic properties of the testing strategy and (iii) provides further Monte Carlo results on the finite sample properties of the testing strategy.

2 Structural model and reduced form solutions

This section presents the reference small-scale new-Keynesian business cycle model, summarize its time series representations under determinacy and indeterminacy, and discusses the conditions which give rise to observational equivalence. This is useful to understand the mechanics of the testing approach presented next.

Our reference new-Keynesian model is taken from Benati and Surico (2009). It features the following three equations:

$$\tilde{y}_t = \gamma E_t \tilde{y}_{t+1} + (1-\gamma)\tilde{y}_{t-1} - \delta(R_t - E_t \pi_{t+1}) + \omega_{\tilde{y},t} \quad (1)$$

$$\pi_t = \frac{\beta}{1+\beta\alpha} E_t \pi_{t+1} + \frac{\alpha}{1+\beta\alpha}\pi_{t-1} + \kappa \tilde{y}_t + \omega_{\pi,t} \quad (2)$$

$$R_t = \rho R_{t-1} + (1-\rho)(\varphi_\pi \pi_t + \varphi_{\tilde{y}} \tilde{y}_t) + \omega_{R,t} \quad (3)$$

where

$$\omega_{x,t} = \rho_x \omega_{x,t-1} + \varepsilon_{x,t} \quad , -1<\rho_x<1 \quad , \varepsilon_{x,t} \sim \text{WN}(0, \sigma_x^2) \quad , \quad x = \tilde{y}, \pi, R \quad (4)$$

and expectations are conditional on the information set $\mathcal{F}_t$, i.e. $E_t \cdot := E(\cdot \mid \mathcal{F}_t)$. The variables $\tilde{y}_t$, π_t, and R_t stand for the output gap, inflation, and the nominal interest rate, respectively; γ is the weight of the forward-looking component in the intertemporal IS curve; α is price setters' extent of indexation to past inflation; δ is households' intertemporal elasticity of substitution; κ is the slope of the Phillips curve; ρ, φ_π, and $\varphi_{\tilde{y}}$ are the interest rate smoothing coefficient, the long-run coefficient on inflation, and that on the output gap in the monetary policy rule, respectively; finally, $\omega_{\tilde{y},t}$, $\omega_{\pi,t}$ and $\omega_{R,t}$ in eq. (4) are the mutually independent, autoregressive of order one disturbances and $\varepsilon_{\tilde{y},t}$, $\varepsilon_{\pi,t}$ and $\varepsilon_{R,t}$ are the structural (fundamental) shocks. This or similar small-scale models have successfully been employed to conduct empirical analysis concerning the U.S. economy. Clarida *et al.* (2000) and Lubik and Schorfheide (2004) have investigated the influence of systematic monetary policy over the U.S. macroeconomic dynamics; Boivin and Giannoni (2006), Benati and Surico (2009), and Lubik and Surico (2010) have replicated the U.S. Great Moderation, Benati (2008) and Benati and Surico (2008) have investigated the drivers of the U.S. inflation persistence; Castelnuovo and

Surico (2010) have replicated the VAR dynamics conditional on a monetary policy shock in different sub-samples; Inoue and Rossi (2011*a*) have analyzed the role of parameter instabilities as drivers of the Great Moderation.

We compact the system composed by eq.s (1)-(4) in the representation

$$\Gamma_0 X_t = \Gamma_f E_t X_{t+1} + \Gamma_b X_{t-1} + \omega_t \tag{5}$$

$$\omega_t = \Xi\omega_{t-1} + \varepsilon_t \ , \ \ \varepsilon_t \sim \text{WN}(0, \Sigma_\varepsilon) \tag{6}$$

$$\Xi{:=}dg(\rho_{\tilde{y}}, \rho_\pi, \rho_R) \ \ , \ \ \Sigma_\varepsilon{:=}dg(\sigma^2_{\tilde{y}}, \sigma^2_\pi, \sigma^2_R)$$

where $X_t{:=}(\tilde{y}_t, \pi_t, R_t)'$, $\omega_t{:=}(\omega_{\tilde{y},t}, \omega_{\pi,t}, \omega_{R,t})'$, $\varepsilon_t{:=}(\varepsilon_{\tilde{y},t}, \varepsilon_{\pi,t}, \varepsilon_{R,t})'$ and

$$\Gamma_0{:=}\begin{pmatrix} 1 & 0 & \delta \\ -\kappa & 1 & 0 \\ -(1-\rho)\varphi_{\tilde{y}} & -(1-\rho)\varphi_\pi & 1 \end{pmatrix}, \Gamma_f{:=}\begin{pmatrix} \gamma & \delta & 0 \\ 0 & \frac{\beta}{1+\beta\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Gamma_b{:=}\begin{pmatrix} 1-\gamma & 0 & 0 \\ 0 & \frac{\alpha}{1+\beta\alpha} & 0 \\ 0 & 0 & \rho \end{pmatrix}.$$

Let $\theta{:=}(\gamma, \delta, \beta, \alpha, \kappa, \rho, \varphi_{\tilde{y}}, \varphi_\pi, \rho_{\tilde{y}}, \rho_\pi, \rho_R, \sigma^2_{\tilde{y}}, \sigma^2_\pi, \sigma^2_R)'$ be the $m \times 1$ vector of structural parameters ($m{:=}\dim(\theta)$). The elements of the matrices Γ_0, Γ_f, Γ_b and Ξ depend nonlinearly on θ and, without loss of generality, the matrix $\Gamma_0^\Xi{:=}(\Gamma_0 + \Xi\Gamma_f)$ is assumed to be non-singular. The space of all theoretically admissible values of θ is denoted by $\mathcal{P}$.

For future uses, we consider the partition $\theta{:=}(\theta_s', \theta_\varepsilon')'$, where θ_ε contains the non-zero elements of $vech(\Sigma_\varepsilon)$ and θ_s all remaining elements. The 'true' value of θ, $\theta_0{:=}(\theta_{0,s}', \theta_{0,\varepsilon}')'$, is assumed to be an interior point of $\mathcal{P}$. Given the partition $\theta{:=}(\theta_s', \theta_\varepsilon')'$, we also consider the corresponding partition of the parameter space $\mathcal{P}{:=}\mathcal{P}_{\theta_s} \times \mathcal{P}_{\theta_\varepsilon}$. This distinction is important for two related reasons. First, the determinacy/indeterminacy of the system depends only on the values taken by θ_s, and not by θ_ε. Second, the sub-vector θ_ε is not directly recoverable (identifiable) from the estimation of the system of Euler equations (5)-(6) through 'limited-information' methods, and our procedure for testing determinacy/indeterminacy also relies on the direct estimation of θ_s from system (5)-(6).

Throughout the paper, we use the notations '$M(\theta)$' and '$M{:=}M(\theta)$' to indicate that the elements of the matrix M depend nonlinearly on the structural parameters θ, hence in our setup $\Gamma_0{:=}\Gamma_0(\theta)$, $\Gamma_f{:=}\Gamma_f(\theta)$, $\Gamma_b{:=}\Gamma_b(\theta)$ and $\Xi{:=}\Xi(\theta)$. Moreover, we call 'stable' a matrix that has all eigenvalues inside the unit disk and 'unstable' a matrix that has at least one eigenvalue outside the unit disk. Thus, denoted with $\lambda_{\max}(\cdot)$ the absolute value of the largest eigenvalue of the matrix in the argument, we have $\lambda_{\max}(M(\theta)) < 1$ for stable matrices and $\lambda_{\max}(M(\theta)) > 1$ for unstable ones.

The solution properties of the system of Euler equations (5)-(6) depend on whether θ_s lies in the determinacy or indeterminacy region of the parameter space. Thus, the theoretically admissible parameter space $\mathcal{P}_{\theta_s}$ is decomposed into two disjoint subspaces, the determinacy region, $\mathcal{P}_{\theta_s}^D$, and its complement $\mathcal{P}_{\theta_s}^I:=\mathcal{P}_{\theta_s}\backslash\mathcal{P}_{\theta_s}^D$. We assume that $\forall\theta_s \in \mathcal{P}_{\theta_s}$, an asymptotically stationary (stable) reduced form solution to system (5)-(6) exists, hence the case of non stationary and 'explosive' (unstable) indeterminacy is automatically ruled out. Since we consider only stationary solutions, $\mathcal{P}_{\theta_s}^I$ contains only values of θ_s that lead to multiple stable solutions. The whole set of regularity conditions assumed to hold in the specified structural system are reported in our supplementary material, where we show that the stability/instability of the matrix $G(\theta_s):=(\Gamma_0^\Xi - \Gamma_f\Phi_1)^{-1}\Gamma_f$, where $\Gamma_0^\Xi:=(\Gamma_0 + \Xi\Gamma_f)$, and Φ_1 is a matrix that will be discussed below, can be associated with the determinacy/indeterminacy of the system.

Determinacy/indeterminacy is a system property that depends on all elements in θ_s. There are cases in which the new-Keynesian system is highly restricted and it becomes relatively simple to identify the region $\mathcal{P}_{\theta_s}^D$ ($\mathcal{P}_{\theta_s}^I$) of the parameter space. For instance, if system (1)-(4) is restricted such that $\gamma:=1$, $\alpha:=0$, and $\rho:=0$, $\rho_x:=0$, $x = \tilde{y}, \pi, R$, the model collapses to a 'purely forward-looking' model. In this particular case, it can be shown that the inequality

$$\varphi_\pi + \frac{1-\beta}{\kappa}\varphi_{\tilde{y}} > 1 \tag{7}$$

is sufficient and 'generically' necessary (Woodford, 2003, Proposition 4.3, p. 254) for determinacy. Consequently, the determinacy region of the parameter space is given by
$\mathcal{P}_{\theta_s}^D:=\left\{\theta_s \in \mathcal{P}_{\theta_s},\, \varphi_\pi + \frac{1-\beta}{\kappa}\varphi_{\tilde{y}} > 1\right\}$. However, it is in general not possible to work out a set of closed-form inequality constraints from system (5)-(6) that are both necessary and sufficient for determinacy (indeterminacy) and that can potentially be used to test whether $\theta_{0,s}$ lies in $\mathcal{P}_{\theta_s}^D$ or $\mathcal{P}_{\theta_s}^I$.[3]

2.1 Time series representation under determinacy

For values of θ_s such that the matrix $G(\theta_s):=(\Gamma_0^\Xi - \Gamma_f\Phi_1)^{-1}\Gamma_f$ is stable, i.e. $\lambda_{\max}(G(\theta_s))<1$, the system (5)-(6) has a unique stable reduced form solution that can be represented as the finite-order VAR

$$(I_3 - \Phi_1(\theta_s)L - \Phi_2(\theta_s)L^2)X_t = u_t \quad , \quad u_t:=\Upsilon(\theta_s)^{-1}\varepsilon_t \tag{8}$$

[3] The following example shows that the condition in eq. (7) is not sufficient for determinacy, if the structural model in eq.s (1)-(4) involves lags of the variables, other than leads. Consider the system based on $\beta:=0.99$, $\kappa:=0.085$, $\delta:=0.40$, $\gamma:=0.25$, $\alpha:=0.05$, $\rho:=0.95$, $\varphi_{\tilde{y}}:=2$, $\varphi_\pi:=0.77$, $\rho_{\tilde{y}}:=\rho_\pi:=\rho_R:=0.9$. In this case, the condition $\varphi_\pi + \frac{1-\beta}{\kappa}\varphi_{\tilde{y}} > 1$ is valid but the rational expectation-solution to system (1)-(4), while being stable, is not unique. Recall that we assume the existence of at least a solution under rational expectations.

where L is the lag/lead operator ($L^h X_t{:=}X_{t-h}$), X_0 and X_{-1} are fixed initial conditions, $\Phi_1(\theta_s)$, $\Phi_2(\theta_s)$ and $\Upsilon(\theta_s)$ are $3{\times}3$ matrices whose elements depend nonlinearly on θ_s and embody the cross-equation restrictions implied by the small new-Keynesian model (Hansen and Sargent, 1980, 1981). As shown in the Supplementary Material, the matrices $\Phi_1(\theta_s)$ and $\Phi_2(\theta_s)$ in eq. (8) are obtained as the unique solution to the second-order quadratic matrix equation

$$\mathring{\Phi}{=}(\mathring{\Gamma}_0 - \mathring{\Gamma}_f\mathring{\Phi})^{-1}\mathring{\Gamma}_b \tag{9}$$

where $\mathring{\Gamma}_f$, $\mathring{\Gamma}_0$, $\mathring{\Gamma}_b$ and the stable matrix $\mathring{\Phi}$ are respectively given by

$$\mathring{\Gamma}_0{:=}\begin{pmatrix} \Gamma_0^{\Xi} & 0_{3\times3} \\ 0_{3\times3} & I_3 \end{pmatrix}, \ \mathring{\Gamma}_f{:=}\begin{pmatrix} \Gamma_f & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{pmatrix}, \ \mathring{\Gamma}_b{:=}\begin{pmatrix} \Gamma_{b,1}^{\Xi} & \Gamma_{b,2}^{\Xi} \\ I_3 & 0_{3\times3} \end{pmatrix}, \ \mathring{\Phi}{:=}\begin{pmatrix} \Phi_1 & \Phi_2 \\ I_3 & 0_{3\times3} \end{pmatrix},$$

and $\Gamma_{b,1}^{\Xi}{:=}(\Gamma_b + \Xi\Gamma_0)$, $\Gamma_{b,2}^{\Xi}{:=}{-}\Xi\Gamma_b$ and $\Upsilon(\theta){:=}(\Gamma_0 - \Gamma_f\Phi_1(\theta))$. The matrix $\Phi_1{:=}\Phi_1(\theta_s)$ is the one entering the definition of $G(\theta_s)$. The constrained covariance matrix of the reduced form disturbances u_t, denoted with $\tilde{\Sigma}_u$, is given by

$$\tilde{\Sigma}_u(\theta){=}\Upsilon(\theta_s)^{-1}\Sigma_\varepsilon(\theta_\varepsilon)\Upsilon(\theta_s)'^{-1}. \tag{10}$$

Equations (9) and (10) define the CER implied by our new-Keynesian structural model on its reduced form solution under determinacy.

2.2 Time series representation under indeterminacy

For values of θ_s such that the matrix $G(\theta_s){:=}(\Gamma_0^{\Xi}{-}\Gamma_f\Phi_1)^{-1}\Gamma_f$ is unstable, i.e. $\lambda_{\max}(G(\theta_s)){>}1$,[4] the class of reduced form solutions associated with the new-Keynesian system (5)-(6) becomes more involved from a dynamic standpoint, see Lubik and Schorfheide (2003, 2004) and Fanelli (2012).

In particular, when $\lambda_{\max}(G(\theta_s)){>}1$, the matrix $G(\theta_s)$ can be decomposed in the form

$$G(\theta_s){=}P(\theta_s)\begin{pmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{pmatrix}P^{-1}(\theta_s)$$

where $P(\theta_s)$ is a 3×3 non-singular matrix, Λ_1 is the $n_1\times n_1$ ($n_1 < 3$) Jordan normal block that collects the eigenvalues of $G(\theta_s)$ that lie inside the unit disk and Λ_2 is the $n_2 \times n_2$ ($n_2 \leq 3$) Jordan normal block that collects the eigenvalues of $G(\theta_s)$ that lie outside the unit disk. Notice that $n_1 + n_2{=}3$, where $n_2{:=}\dim(\Lambda_2)$ determines the 'degree of multiplicity' of solutions, see

[4] The case in which the matrix $G(\theta_s)$ has eigenvalues equal to one is deliberately ignored because it can be associated with the case of non-stationary processes.

below. In the Supplementary Material we prove that in this case the reduced form solutions can be given the VARMA-type representation:

$$(I_3-\Pi(\theta_s)L)(I_3-\Phi_1(\theta_s)L-\Phi_2(\theta_s)L^2)X_t = (M(\theta_s,\psi)-\Pi(\theta_s)L)V(\theta_s,\psi)^{-1}\varepsilon_t+\tau_t \quad (11)$$

$$\tau_t{:=}(M(\theta_s,\psi) - \Pi(\theta_s)L)V(\theta_s,\psi)^{-1}P(\theta_s)\zeta_t + P(\theta_s)\zeta_t. \quad (12)$$

In this system, the matrices $\Phi_1(\theta_s)$ and $\Phi_2(\theta_s)$ are defined as in the case of determinacy, see eq. (9), while the matrices $\Pi(\theta_s)$, $M(\theta_s,\psi)$ and $V(\theta_s,\psi)$ are given by

$$\Pi(\theta_s){:=}P(\theta_s)\begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{pmatrix} P^{-1}(\theta_s) \quad , \quad M(\theta_s,\psi){:=}P(\theta_s)\begin{pmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Psi \end{pmatrix} P^{-1}(\theta_s)$$

$$V(\theta_s,\psi){:=}(\Gamma_0(\theta_s) - \Gamma_f(\theta_s)\Phi_1(\theta_s)) - \Xi(\theta_s)\Gamma_f(\theta_s)(I_3 - M(\theta_s,\psi)).$$

In this framework, Ψ is a $n_2\times n_2$ matrix ($n_2 \leq 3$) containing a set of arbitrary auxiliary parameters unrelated to θ_s. We collect these parameters in the vector $\psi{:=}vec(\Psi)$. The 'additional' moving average term τ_t which enters system (11)-(12) depends on a 3×1 martingale difference sequence (MDS) vector ζ_t which collects the 'sunspot shocks', and may be unrelated to the fundamental shocks. We assume ζ_t has a time-invariant covariance matrix Σ_ζ. The specific features of the ζ_t component are discussed in detail in the Supplementary Material.

While the determinate equilibrium in eq. (8) depends only on the state variables of the structural system (5)-(6), there are two sources of indeterminacy featured by the equilibria in eq.s (11)-(12). First, there is the 'parametric indeterminacy' that springs from the auxiliary parameters in the vector ψ. Such parameters index solution multiplicity and may amplify or dampen the fluctuations of X_t governed by the VMA part of the reduced form solution. Second, there is the 'stochastic indeterminacy' that stems from the term τ_t, which in turn depends on the sunspot shocks ζ_t (when $\Sigma_\zeta \neq 0_{3\times 3}$). These shocks may arbitrarily alter the dynamics and volatility of the new-Keynesian system, see Lubik and Schorfheide (2003, 2004) and Lubik and Surico (2009) for discussions.

2.3 Observational equivalence

The structure of the two reduced form solutions reported above reveals that, under indeterminacy, the parameter space associated with the new-Keynesian model is wider compared to the case of determinacy. Indeed, in addition to the structural parameters θ, the dynamics of the system is also governed by

ψ and σ_{ζ}^{+}, where σ_{ζ}^{+} collects the free elements of the covariance matrix Σ_{ζ}. Both ψ and σ_{ζ}^{+} are unrelated to θ and are not identified under determinacy.

Let $\mathcal{N}$ be the open sub-space of $\mathbb{R}^{(n_2)^2}$ of all possible values taken by ψ, and let $\mathcal{Z}$ be the open sub-space of $\mathbb{R}^6$ of all possible values taken by the elements in σ_{ζ}^{+}; the 'complete' parameter space associated with indeterminacy is[5]

$$\mathcal{I}{:=}\left\{\theta^{*}{:=}(\theta', \psi', \sigma_{\zeta}^{+\prime})', \, \theta_s \in \mathcal{P}_{\theta_s}^{I}, \, \psi \in \mathcal{N}, \, \sigma_{\zeta}^{+} \in \mathcal{Z}\right\}. \tag{13}$$

In the special case in which ψ and σ_{ζ}^{+} fulfil the conditions

$$\psi{=}vec(I_{(n_2)^2}) \quad (\Rightarrow M(\theta_s, \psi){=}I_3) \quad , \quad \sigma_{\zeta}^{+}{=}0_{6\times 1} \quad (\Rightarrow \tau_t{=}0_{3\times 1} \text{ a.s. } \forall \, t), \tag{14}$$

system (11)-(12) collapses to a MSV solution (McCallum, 1983), i.e., a reduced form solution which has the same time series representation as the determinate VAR solution in eq. (8), and it is subject to the same set of cross-equation restrictions, see Evans and Honkapohja (1986), Lubik and Schorfheide (2003, 2004), and Fanelli (2012).[6] This observational equivalence reflects on the properties of the testing strategy we present below.

3 Inferential issues

Let X_1, ..., X_T be a sample of T observations that are supposed to be generated by a solution of the new-Keynesian system (5)-(6). Our task is to decide whether the observations X_1, ..., X_T are consistent with the case of a unique stable equilibrium, or the case of multiple stable equilibria, controlling for two factors: (i) the possible identification failures, where by this term we denote the case in which the objective functions used to estimate the structural

[5] For given a $\theta_s \in \mathcal{P}_{\theta_s}^{I}$, the auxiliary parameters ψ might in principle lie in a region of $\mathcal{N}$ such that the VMA components of system (11)-(12) are non-invertible. Under this scenario, the possibility of recovering the structural shocks from the history of X_t is compromised even when the econometrician can observe all components of X_t. Thus, indeterminacy can be a further source of 'non-fundamentalness' in business cycle analysis.

[6] Observational equivalence between determinate and indeterminate reduced form solutions may be also obtained from system (5) when the vector of fundamental shocks is absent, i.e. when $\Sigma_{\varepsilon}{=}0_{3\times 3}$ ($\varepsilon_t{=}0_{3\times 1}$ a.s. $\forall$ t). In this case, under a set of restrictions, including $\Xi{=}0_{n\times n}$, the structural model can be solved and represented as in eq. (8). Thus, there exists an intrinsic identification problem once we consider also 'exact' DSGE models: an indeterminate equilibrium of an 'exact' model (i.e. featuring $\varepsilon_t{=}0_{3\times 1}$ and $\Xi{=}0_{n\times n}$), can be observationally equivalent to the determinate equilibrium of an DSGE model with $\varepsilon_t \neq 0_{3\times 1}$ but richer dynamic structure, see Beyer and Farmer (2007) and Fanelli (2012) for a comprehensive discussion. While being interesting from a theoretical standpoint, the case of absence of fundamental shocks in the structural equations is empirically unpalatable, and it will not be considered in our analysis.

parameters and derive the test statistics may be poorly informative about θ or some of its components; (ii) the possible 'dynamic misspecification', where by this term we denote the situation in which the system (5)-(6) omits relevant propagation mechanisms.

An ideal test for the null $H_0 : \theta_{0,s} \in \mathcal{P}^D_{\theta_s}$ against the alternative $H_1 : \theta_{0,s} \in \mathcal{P}^I_{\theta_s}$ should be based on testing the set of inequality restrictions that identify the region $\mathcal{P}^D_{\theta_s}$ ($\mathcal{P}^I_{\theta_s}$) of the parameter space. For instance, Mavroeidis (2010) uses the standard 'Taylor principle' condition in eq. (7) to address the determinacy/indeterminacy issue in U.S. monetary policy by estimating a Taylor-type monetary policy rule in isolation from other structural equations. The typical risk with this 'single-equation' approach is that the 'Taylor principle' holds with certainty in the form of eq. (7) only if the structural system (5)-(6) fulfills e.g. the restrictions γ:=1, α:=0, and ρ:=0, ρ_x:=0, $x = \tilde{y}, \pi, R$. Our estimates reported in Section 5 show that these restrictions are invalid. In our framework, a 'generic' characterization of the indeterminacy region of the parameter space $\mathcal{P}^I_{\theta_s}$ is given by $\mathcal{P}^I_{\theta_s}:=\{\theta_s \in \mathcal{P}_{\theta_s},\, \lambda_{\max}(G(\theta_s))>1\}$, see Section 2 and the Supplementary Material. Unfortunately, even under strong identification, the condition $\lambda_{\max}(G(\theta_s))>1$ can hardly be used for testing purposes. Indeed, aside from very special cases, it is not easy to map the inequalities restrictions that characterize the unstable eigenvalues of the $G(\theta_s)$ matrix onto a set of 'manageable' restrictions on the elements of θ_s.[7] Even working out the inequalities associated with the condition $\lambda_{\max}(G(\theta_s))>1$ on a case-by-case basis, the resulting testing problem would involve nonstandard inference, see e.g. Silvapulle and Sen (2005) and references therein.

To circumvent the above mentioned difficulties, we address the testing problem from another viewpoint. We consider the following hypotheses:

$$H_0' : X_t \text{ is generated by the VAR system (8) under the CER in eq.s (9)-(10)} \quad (15)$$

$$H_1' : X_t \text{ is generated by the VARMA-type system (11)-(12), with } \theta^* \in \mathcal{I}^0 \quad (16)$$

where $\mathcal{I}^0$ is a subset of $\mathcal{I}$ in eq. (13), defined by

$$\mathcal{I}^0 := \left\{\theta^* := (\theta', \psi', \sigma_\zeta^{+\prime})',\, \theta_s \in \mathcal{P}^I_{\theta_s},\, \psi \in \mathcal{N} \setminus \left\{vec(I_{(n_2)^2})\right\},\, \sigma_\zeta^+ \in \mathcal{Z} \setminus \{0_{6\times 1}\}\right\} \subset \mathcal{I}. \quad (17)$$

Under H_0', the new-Keynesian system admits the same time series representation as the unique stable solution but is observationally indistinguishable from the indeterminate MSV equilibrium obtained from the system (11)-(12)

[7] Farmer and Guo (1995) use the inequality restriction that identify the indeterminacy region of the parameter space in their stylized business cycle model, and show that their point estimates of the structural parameters fulfil the restriction. However, no inference is provided is such paper.

when ψ and σ_{ζ}^{+} satisfy the conditions in eq. (14). Under H_1', instead, the new-Keynesian system generates indeterminate non-MSV equilibria. A key observation is that the null of determinacy, H_0: $\theta_{0,s} \in \mathcal{P}_{\theta_s}^{D}$, implies the hypothesis H_0', while the converse is not true. Hence, the rejection of H_0' is evidence against determinacy, while the non-rejection of H_0' can not be considered conclusive evidence of determinacy. Indeed, the non-rejection of H_0' is only sufficient to rule out the case of 'parametric indeterminacy' generated by the presence of the auxiliary parameters ψ, and the 'stochastic indeterminacy' generated by the sunspot shocks ($\sigma_{\zeta}^{+} \neq 0_{6\times 1}$), but is not sufficient to rule out the case of a MSV solution nested in the class of models in eq.s (11)-(12).

To build our identification-robust test for H_0' against H_1', we exploit the well known fact that the construction of confidence sets is a dual problem to hypothesis testing, i.e. confidence sets are obtained by inverting tests, see e.g. Aitchison (1964).[8] In turn, inverting a test means considering all parameter values that are not rejected by the test at a pre-fixed significance level. Our robust testing strategy combines the information deriving from two types of identification-robust inferential approaches. The former, presented in Sub-section 3.1, is a 'full-information' identification-robust approach which allow us to build a confidence set for θ_s exploiting the CER implied by the new-Keynesian system under determinacy. The latter, summarized in Sub-section 3.2, is a 'limited-information' identification-robust approach which allow us to build a confidence set for θ_s exploiting the OR implied by the new-Keynesian system system under the rational expectations hypothesis. These two methods are condensed in Sub-section 3.3 in a coherent testing strategy for H_0' against H_1'.

3.1 Full-information approach for the CER

We consider the reduced form finite-order VAR solution of the new-Keynesian model in eq. (8), and the vector of reduced form coefficients ϕ:=$(\phi^{*\prime}, vech(\Sigma_u)')'$, where ϕ^{*}:=$vec(\Phi_u)$, and the matrix Φ_u:=$[\Phi_1, \Phi_2]$ collects the VAR coefficients. In our setup, ϕ is assumed to be strongly identified. This assumption valid when all components of X_t are observed. For cases in which X_t features unobserved components, it is necessary to refer to the minimal state-space representation associated with the new-Keynesian system under determinacy on a model-by-model basis, see Komunjer and Ng (2011) and

[8] This approach has been used in the recent literature on inference in weakly identified DSGE models, see Dufour *et al.* (2009, 2013), Kleibergen and Mavroeidis (2009), Mavroeidis (2010), Qu (2011), Andrews and Mikusheva (2012) and Guerron-Quintana *et al.* (2013).

Guerron-Quintana *et al.* (2013) for examples and discussion. We denote with $\log L_T(\phi)$ the log-likelihood function associated with the finite-order VAR in eq. (8) before imposing the CER.

The CER that the new-Keynesian model places on its determinate reduced form solution in eq.s (9)-(10) can conveniently be compacted in the expression

$$f(\phi,\theta) = 0_{\dim(\phi)\times 1} \tag{18}$$

where $f(\cdot,\cdot)$ is a continuous, twice differentiable, vector function. By the implicit function theorem, the restrictions in eq. (18) can also be written in explicit form as follows (see Iskrev, 2008):

$$\phi{=}g(\theta) \tag{19}$$

where $g(\cdot)$ is a nonlinear twice differentiable function and the mapping from θ to ϕ is valid in a neighborhood of the true parameter values. Under the CER in eq. (19), the VAR log-likelihood depends on θ and is denoted with $\log L_T(g(\theta))$. In our setup, the shape of $\log L_T(g(\theta))$ may be poorly informative (or uninformative) about θ or some of its components, violating one of the standard regularity conditions behind ML estimation, see, *inter alia*, Andrews and Mikusheva (2012). Throughout the paper we maintain that θ_ε in $\theta{:=}(\theta_s', \theta_\varepsilon')'$ is strongly identified, and that identification failure may solely affect θ_s or some of its components. This assumption reflects the situation we typically observe in practice, where weak identification or unidentification typically involve the elements in θ_s and not the elements in θ_ε.[9] Under this assumption, for any given value of $\theta_s{=}\breve{\theta}_s \in \mathcal{P}_{\theta_s}$, the log-likelihood function $\log L_T(g(\breve{\theta}_s, \theta_\varepsilon))$ depends on θ_ε alone, and fulfills the regularity conditions discussed in e.g. Guerron-Quintana *et al.* (2013).

Keeping these observations in mind, we face the problem of computing a LR test for the null hypothesis that there exists a θ_ε such that

$$H_{0,cer}\colon\ \phi_{\breve{\theta}_s}{=}g(\breve{\theta}_s, \theta_\varepsilon)\ \ ,\ \ \theta_s{=}\breve{\theta}_s \in \mathcal{P}_{\theta_s} \tag{20}$$

(against the alternative $H_{1,cer}:\ \phi_{\breve{\theta}_s} \neq g(\breve{\theta}_s, \theta_\varepsilon)$). The hypothesis $H_{0,cer}$ in eq. (20) is composite: it specializes the CER in eq. (19) to the 'guess' $\theta_s{=}\breve{\theta}_s$ about the parameters value. The notation '$\phi_{\breve{\theta}_s}$' used in eq. (19) remarks that under the CER, the VAR reduced form coefficients depend on the choice $\theta_s{=}\breve{\theta}_s$. When $H_{0,cer}$ is valid, also the hypothesis H_0' in eq. (15) is valid for $\theta_s{=}\breve{\theta}_s$. Likewise, when H_0' in eq. (15) is valid for some $\theta_s{=}\breve{\theta}_s$, the hypothesis

[9] This assumption might be relaxed without changing the logic of the proposed testing strategy.

$H_{0,cer}$ in eq. (20) will be automatically valid. However, while H_0' is accepted if there exists at least one $\theta_s=\breve{\theta}_s$ such that $H_{0,cer}$ holds, it is rejected if and only if $H_{0,cer}$ is rejected for all possible parameter values.

Let $LR_T(\hat{\phi}_{\breve{\theta}_s}):=-2(\log L_T(\hat{\phi}_{\breve{\theta}_s}) - \log L_T(\hat{\phi}))$ be the likelihood-ratio test statistic for the hypothesis $H_{0,cer}$, where the vector $\hat{\phi}_{\breve{\theta}_s}$ is defined by $\hat{\phi}_{\breve{\theta}_s}:=g(\breve{\theta}_s, \hat{\theta}_\varepsilon^{\breve{\theta}_s})$, and $\hat{\theta}_\varepsilon^{\breve{\theta}_s}$ is the ML estimate of θ_ε obtained for $\theta_s=\breve{\theta}_s$. Under $H_{0,cer}$, the asymptotic null distribution of $LR_T(\hat{\phi}_{\breve{\theta}_s})$ is pivotal and $\chi^2_{d_1}$, where $d_1:=\dim(\phi)-\dim(\theta_\varepsilon)$, regardless of whether θ_s is identified or not, see e.g. Guerron-Quintana *et al.* (2013). In practice, there might be many possible choices $\theta_s=\breve{\theta}_s$ not rejected by the test $LR_T(\hat{\phi}_{\breve{\theta}_s})$. Since the components of θ_s typically lie within bounded (theoretically admissible) intervals, one can test $H_{0,cer}$ for many possible choices of $\breve{\theta}_s$ within a fine grid $\mathcal{G}_{\theta_s}$ in $\mathcal{P}_{\theta_s}$, giving rise to a 'grid testing' procedure. The numerical inversion of the test $LR_T(\hat{\phi}_{\breve{\theta}_s})$ for $H_{0,cer}$ gives rise to the identification-robust confidence set (or acceptance region of the test):

$$\mathcal{C}^{LR}_{1-\eta_1}:=\left\{\breve{\theta}_s \in \mathcal{G}_{\theta_s},\ LR_T(\hat{\phi}_{\breve{\theta}_s}) < c^{\eta_1}_{\chi^2_{d_1}}\right\} \tag{21}$$

where $c^{\eta_1}_{\chi^2_{d_1}}$ is the η_1-level cut-off point associated with the $\chi^2_{d_1}$ distribution, and $0<\eta_1<1$ is the pre-fixed nominal level of significance (or type-I error) of the test.[10] The identification-robust confidence set $\mathcal{C}^{LR}_{1-\eta_1}$ has asymptotic coverage $100(1-\eta_1)$ (see Supplementary Material). A point estimate of θ_s can be obtained from the (nonempty) confidence set $\mathcal{C}^{LR}_{1-\eta_1}$ by

$$\hat{\theta}_{s,ML} := \underset{\breve{\theta}_s\in\mathcal{C}^{LR}_{1-\eta_1}}{\arg\min}\ LR_T(\hat{\phi}_{\breve{\theta}_s}), \tag{22}$$

i.e. considering the parameter point within $\mathcal{C}^{LR}_{1-\eta_1}$ with associated largest p-value (or the 'least rejected' models at the pre-fixed level η_1).[11]

The identification-robust confidence set $\mathcal{C}^{LR}_{1-\eta_1}$ in eq. (21) is built in a 'full-information' framework, in the sense that inverting the test for the null in eq. (20) requires computing the determinate rational expectations solution associated with the new-Keynesian system.

[10] Dufour *et al.* (2013) have proposed another identification-robust 'full-information' approach for the structural parameters of DSGE models based on the (numerical) inversion of a test for zero coefficients in the multivariate regression of the quantity $u_t(\breve{\theta}_s):=(I_3-\Phi_1(\breve{\theta}_s)L-\Phi_2(\breve{\theta}_s)L^2)X_t$ (which correspond to the disturbance of the VAR system (8) under the CER) on the regressors $Z_t:=(X_{t-1}', X_{t-2}')'$.

[11] The point estimates in eq. (22) can be interpreted as 'Hodges-Lehmann' estimates of θ_s, see e.g. Dufour *et al.* (2006, 2009, 2010).

3.2 Limited-information approach for the system of structural Euler equations

We now focus on the system of Euler equations (5)-(6), and consider the problem of testing the simple hypothesis

$$H_{0,spec}\colon\ \theta_s{=}\breve{\theta}_s \quad , \ \ \breve{\theta}_s \in \mathcal{P}_{\theta_s} \tag{23}$$

against the alternative $H_{1,spec}$: $\theta_s \neq \breve{\theta}_s$, abstracting from the knowledge of the reduced form solution of the model. $H_{0,spec}$ is the hypothesis that the system of Euler equations (5)-(6) is valid in correspondence of the 'guess' $\theta_s{=}\breve{\theta}_s$ about the parameters value.

Following Dufour *et al.* (2013), a test for $H_{0,spec}$ can be computed as follows. By simple algebraic manipulations, we re-write system (5)-(6) in the form

$$\Gamma_0^{\Xi} X_t = \Gamma_f X_{t+1} + \Gamma_{b,1}^{\Xi} X_{t-1} + \Gamma_{b,2}^{\Xi} X_{t-2} + \Xi\Gamma_f \xi_t + \varepsilon_t - \Gamma_f \xi_{t+1},$$

where $\xi_t{:=}X_t - E_{t-1}X_t$ is a vector MDS, and then define the 3×1 vector function

$$v(X_t, \theta_s){:=}\Gamma_0^{\Xi} X_t - \Gamma_f X_{t+1} - \Gamma_{b,1}^{\Xi} X_{t-1} - \Gamma_{b,2}^{\Xi} X_{t-2} = \Xi\Gamma_f \xi_t + \varepsilon_t - \Gamma_f \xi_{t+1}. \tag{24}$$

Under $H_{0,spec}$, the term $v(X_t, \breve{\theta}_s)$ follows a VMA(1)-type process and fulfills the OR:

$$E\left(v(X_t, \breve{\theta}_s) \mid \mathcal{F}_{t-1}\right) = 0_{3\times 1}. \tag{25}$$

Therefore, we can associate the multivariate linear regression model:

$$v(X_t, \breve{\theta}_s) = \Pi_{\breve{\theta}_s} Z_t + \epsilon_t \quad , \quad Z_t \in \mathcal{F}_{t-1} \quad , \ t = 1, ..., T \tag{26}$$

to the hypothesis $H_{0,spec}$. In eq. (26), $\Pi_{\breve{\theta}_s}$ is a $3{\times}r$ matrix of coefficients, Z_t is a $r \times 1$ vector of regressors selected from the information set $\mathcal{F}_{t-1}$, and ϵ_t is a disturbance term whose covariance matrix, Σ_ϵ, can possibly be non-diagonal. The notation '$\Pi_{\breve{\theta}_s}$' for the regression coefficients remarks that we have a multivariate regression system like that in eq. (26) for each choice $\theta_s{=}\breve{\theta}_s$. Under $H_{0,spec}$, additional information from predetermined variables should be irrelevant, hence the associated problem

$$H_{0,spec}^{*} : \Pi_{\breve{\theta}_s}{=}0_{3\times r} \ \text{vs} \ \ H_{1,spec}^{*} : \Pi_{\breve{\theta}_s} \neq 0_{3\times r} \tag{27}$$

should lead us to accept $H_{0,spec}^{*}$. We have thus transformed the problem of testing the hypothesis $H_{0,spec}$ (against $H_{1,spec}$) into the problem of testing the hypothesis $H^*_{0,spec}$ (against $H^*_{1,spec}$) in the multivariate linear regression system (26). Standard asymptotic theory applies for the testing problem in eq. (27) irrespective of whether θ_s is identifiable or not.

Let $AR_T(\breve{\theta}_s)$ be any asymptotically pivotal test statistic for the problem in eq. (27). In practice, $AR_T(\breve{\theta}_s)$ can be a Wald-type, a Lagrange Multiplier or (quasi-)LR test, and can be interpreted as an Anderson Rubin-type test (Anderson and Rubin, 1949).[12] Under $H_{0,spec}$, the asymptotic null distribution of $AR_T(\breve{\theta}_s)$ is $\chi^2_{d_2}$, $d_2{:=}3r$ and also in this case there might be many choices $\theta_s{=}\breve{\theta}_s$ not rejected by the test $AR_T(\breve{\theta}_s)$. The numerical inversion of the test $AR_T(\breve{\theta}_s)$ for $H_{0,spec}$ leads to the identification-robust confidence set (or acceptance region):

$$\mathcal{C}^{AR}_{1-\eta_2}{:=}\left\{\breve{\theta}_s \in \mathcal{D}_{\theta_s},\, AR_T(\breve{\theta}_s) < c^{\eta_2}_{\chi^2_{d_2}}\right\} \tag{28}$$

where $\mathcal{D}_{\theta_s}$ is a fine grid in $\mathcal{P}_{\theta_s}$, $c^{\eta_2}_{\chi^2_{d_2}}$ is the η_2-level cut-off point associated with the $\chi^2_{d_2}$ distribution, and $0{<}\eta_2{<}1$ is the pre-fixed nominal level of significance (or type-I error) of the test. The identification-robust confidence set $\mathcal{C}^{AR}_{1-\eta_2}$ has asymptotic coverage $100(1{-}\eta_2)$ (see Supplementary Material) and defines the set of parameter points in $\mathcal{P}_{\theta_s}$ which are consistent with the new-Keynesian model at the significance level η_2 regardless of the multiplicity/uniqueness of its solutions. A point estimate of θ_s can be obtained from the (nonempty) confidence set $\mathcal{C}^{AR}_{1-\eta_2}$ by

$$\hat{\theta}_{s,LI} := \underset{\breve{\theta}_s \in \mathcal{C}^{AR}_{1-\eta_2}}{\arg\min}\; AR_T(\breve{\theta}_s). \tag{29}$$

It is worth observing that both methods discussed in this and in the previous sub-section refer to estimation of the full system of equations. However, while the 'full-information' method presented in Sub-section 3.1 imposes the additional restriction that the reduced form is a finite-order VAR and exploits the CER implied by the structural model, the 'limited-information' approach summarized here ignores, by construction, any information stemming from the reduced form solutions. Mavroeidis *et al.* (2014), Section 3, discuss the

[12] Since the ϵ_t term follows a VMA-type process in system (26), HAC-type versions of the tests can be applied as suggested by Dufour *et al.* (2013). Alternatively, one can use the 'S-test' method by Stock and Wright (2000), or the 'K-LM test' by Kleibergen (2005), both based on the evaluation of the criterion function corresponding to the continuos-updating version of generalized method of moments. Some computational issues make us prefer the approach in Dufour *et al.* (2009, 2013). Kleibergen and Mavroeidis (2009) discuss weak instrument robust statistics for testing hypotheses on θ_s or its subset in the GMM framework, and then apply these methods to the new-Keynesian Phillips curve.

difference between the two approaches in the context of a single structural equation.

3.3 Testing strategy

The two estimation/testing methods discussed in the previous sub-sections form the basis of our identification-robust testing strategy for H_0' in eq. (15) against H_1' in eq. (16).

Our approach is based on the following two steps:

Step 1: LR test for the CER. Invert the test $LR_T(\hat{\phi}_{\breve{\theta}_s})$ for $H_{0,cer}$ discussed in Sub-section 3.1 at the level η_1, considering points $\theta_s{=}\breve{\theta}_s$ taken from a fine grid $\mathcal{G}^*_{\theta_s}$ in $\mathcal{P}_{\theta_s}$. This yields the identification-robust confidence set

$$\mathcal{C}^{*LR}_{1-\eta_1}:=\left\{\breve{\theta}_s \in \mathcal{G}^*_{\theta_s},\ LR_T(\hat{\phi}_{\breve{\theta}_s}) < c^{\eta_1}_{\chi^2_{d_1}}\right\} \tag{30}$$

whose asymptotic coverage is at least $1-\eta_1$ (see Supplementary Material). If $\mathcal{C}^{*LR}_{1-\eta_1}$ is nonempty, the null H_0' is accepted and the analysis is stopped. If instead $\mathcal{C}^{*LR}_{1-\eta_1}$ is empty, i.e. the hypothesis $H_{0,cer}$ is rejected for all possible parameter values in the grid implying the rejection of H_0', we move to the next step.

Step 2: Anderson-Rubin test for the OR. Conditional on the confidence set $\mathcal{C}^{LR}_{1-\eta_1}$ being empty, we invert the test $AR_T(\breve{\theta}_s)$ for $H_{0,spec}$ discussed in Sub-section 3.2 at the level η_2, considering points $\theta_s{=}\breve{\theta}_s$ taken from a fine grid $\mathcal{D}^*_{\theta_s}$ such that $\mathcal{D}^*_{\theta_s}:=\left\{\breve{\theta}_s \in \mathcal{P}_{\theta_s},\ \lambda_{\max}(G(\breve{\theta}_s)){>}1\right\}$. This yields the identification-robust confidence set

$$\mathcal{C}^{*AR}_{1-\eta_2}:=\left\{\breve{\theta}_s \in \mathcal{D}^*_{\theta_s},\ AR_T(\breve{\theta}_s) < c^{\eta_2}_{\chi^2_{d_2}}\right\} \tag{31}$$

whose asymptotic coverage is at least $1-\eta_2$ (see Supplementary Material). If $\mathcal{C}^{*AR}_{1-\eta_2}$ is nonempty, we accept the hypothesis H_1' in eq. (16). If instead $\mathcal{C}^{*AR}_{1-\eta_2}$ is empty, i.e. $H_{0,spec}$ is rejected for all possible parameter values in the grid, we reject H_1' and conclude that the new-Keynesian system (5)-(6) omits relevant propagation mechanisms.

Hereafter, we conventionally denote the testing strategy obtained by combining the two steps described above with the symbol '$LR_T \rightarrow AR_T$'. Several remarks are in order.

Remark 1. The idea underlying the '$LR_T \to AR_T$' approach is that if the identification-robust confidence set $\mathcal{C}^{*LR}_{1-\eta_1}$ computed in the first-step is nonempty, there exists at least one point in the parameter space consistent with H_0'. This means that the time series representation of the new-Keynesian model summarized in eq.s (8)-(10) is supported by the data for some θ. If instead the identification-robust confidence set $\mathcal{C}^{*LR}_{1-\eta_1}$ is empty, H_0' is rejected and a second-step is run to decide between H_1' and the dynamic misspecification of the structural new-Keynesian system (5)-(6). The second-step is therefore run conditionally on the rejection of the CER in the first-step. If the identification-robust confidence set $\mathcal{C}^{*AR}_{1-\eta_2}$ computed in the second-step is nonempty, there exists at least one θ in the parameter space consistent with H_1'. Finally, when both $\mathcal{C}^{*LR}_{1-\eta_1}$ and $\mathcal{C}^{*AR}_{1-\eta_2}$ are empty, the new-Keynesian system omits relevant propagation mechanisms and is rejected.

Remark 2. The procedure is asymptotically valid irrespective of the strength of identification, hence it can be applied also when θ is strongly identified. Notably, it does not require the identification of the set of parametric inequality restrictions that define the sub-regions $\mathcal{P}^D_{\theta_s}$ ($\mathcal{P}^I_{\theta_s}$) of the parameter space. The practitioner is therefore not committed to the use of nonstandard asymptotic inference. Moreover, it is not necessary to specify prior distributions for θ and the auxiliary parameters ψ (and σ^+_ζ) that govern solution multiplicity in eq.s (11)-(12). In this respect, the suggested approach can be regarded as an identification-robust alternative to the test proposed by Fanelli (2012) assuming strongly identified models.

Remark 3. Many NK-DSGE models feature unobserved states and reliable proxies for these states are not always available. In these situations, we can still compute the LR test in the first-step along the lines suggested by Guerron-Quintana *et al.* (2013), but the implementation of the Anderson Rubin-type test in the second-step may become problematic. Thus, if LR test for the CER rejects H_0' in the first-step, then it is not possible to decide whether the rejection is due to the occurrence of multiple equilibria (H_1'), or to the omission of relevant propagation mechanisms. The extension of the '$LR_T \to AR_T$' testing strategy towards this direction is the subject for future research.

Remark 4. The hypothesis of no dynamic specification of the NK-DSGE model is given by the composite hypothesis $H^* = H_0' \vee H_1'$. In the Supplementary Material we prove that as a test for H^*, the '$LR_T \to AR_T$' sequential procedure has significance level which is bounded above by the maximum of the nominal type-I errors used for the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test in the first-step and the the $AR_T(\breve{\theta}_s)$ test in the second-step. Thus, if e.g. $\eta_1 = \eta_2$:=0.10, the significance level of the procedure as a test for H^* is asymptotically no larger than 10%.

4 Monte Carlo simulations

In this section, we use Benati and Surico's (2009) new-Keynesian system in eq.s (1)-(4) to investigate the finite sample size performance of the '$LR_T \rightarrow AR_T$' testing strategy through some Monte Carlo experiments. Further Monte Carlo results about the rejection frequency of the testing strategy under indeterminate equilibria that belong to the class of models defined by H_1', and the case of 'dynamic misspecification' are confined in the Supplementary Material.

It is worth noting that we work with a 'semi-structural' expression for the NKPC in eq. (2). Such expression features a slope parameter, κ. According to the new-Keynesian theory of the business cycle, κ is a composite parameter influenced by the Calvo-price stickiness parameter, the discount factor, households' risk aversion, and the elasticity of labor. Identification issues are likely to be (even) more severe when referring to such a 'fully-microfounded' version of the NKPC, see Fukaĉ and Pagan (2006, p.17). Our focus on eq. (2) is justified by our willingness to work with a representative version of the NKPC. This is intended to maximize the comparability of our results to the vast literature dealing with specifications similar to ours.[13]

Artificial data sets are generated from the reduced form solutions discussed in Section 2. In all experiments, we consider $M = 1,000$ replications and samples of length $T = 100$ (not including initial lags). The chosen sample size corresponds roughly to the number of quarterly observations we consider for the 'pre-Volcker' (1954q1-1979q2) and 'Great Moderation' (1985q1-2008q2) samples in the empirical section (see Section 5). For each generated data set, we treat the output gap as observable, reproducing the situation we face in Section 5.

To evaluate the empirical size of the '$LR_T \rightarrow AR_T$' test for the hypothesis H_0', the Monte Carlo design is calibrated to match the model estimated by Benati and Surico (2009) using U.S. data with Bayesian methods. The

[13] The same choice is adopted by e.g. Mavroeidis *et al.* (2014) in their recent review of the NKPC empirical literature. Moreover, severe identification issues affect even the 'semi-structural' version of the NKPC we focus on (at least in the widely adopted uni-equational context), as documented and discussed by, among others, Kleibergen and Mavroeidis (2009) and Mavroeidis *et al.* (2014). Hence, while not fully exploiting the restrictions coming from the theory, our version of the NKPC and the chosen new-Keynesian system in general, represents an interesting data generating process to investigate the properties of the proposed identification-robust testing strategy.

discount factor β:=0.99 is treated as known and estimation involves 13 free parameters, 10 of which are collected in the sub-vector θ_s, and 3 in the sub-vector θ_ε. The true vector of parameters $\theta_0:=(\theta'_{0,s},\theta'_{0,\varepsilon})'$ is calibrated to the medians of the 90% coverage percentiles of the posterior distribution reported in Table 1 of Benati and Surico (2009) (see the 'After the Volcker stabilization' column). The data are generated from the reduced form VAR solution in eq. (8) subject to the CER in eq.s (9)-(10), using a Gaussian distribution for the structural shocks ε_t and a diagonal covariance matrix Σ_ε (hence the elements of the sub-vector $\theta_{0,\varepsilon}$ correspond to the diagonal components of Σ_ε). With this calibration, $\lambda_{\max}(G(\theta_{0,s}))$=0.964.

The numerical inversion of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test (first-step) is obtained on each simulated dataset by using a grid of points described in detail in Table 1. We refer to Andrews and Mikusheva (2014) for practical details about the implementation of grid-testing methods.[14] The log-likelihood maximization algorithm under the CER is adapted from the grid-search numerical method discussed in Bårdsen and Fanelli (2014). The empirical size of the test for H'_0 is evaluated by fixing the type-I error of the test at the level η_1=0.10. The results are reported in Table 1, where we summarize the rejection frequency of the $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ test and the average point estimates of the structural parameters (along with the corresponding Monte Carlo standard errors) obtained from the problem in eq. (22) by replacing $\mathcal{C}^{LR}_{1-\eta_1}$ with $\mathcal{C}^{*LR}_{0.90}$. For completeness, Table 1 also reports the empirical size of the $LR_T(\hat{\phi}_{\theta_{0,s}})$ test for the hypothesis $H_{0,cer}$ in eq. (20) evaluated at the specific point $\breve{\theta}_s=\theta_{s,0}$, see the Supplementary Material for details.

The inverted $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test for H'_0 tends to be slightly conservative, as the empirical size is 7.9% as opposed to the nominal size of 10% (instead the empirical size of the test $LR_T(\hat{\phi}_{\theta_{0,s}})$ for $H_{0,cer}$: $\phi_{\breve{\theta}_s}$=$g(\breve{\theta}_s,\theta_\varepsilon)$, $\breve{\theta}_s=\theta_{s,0}$ is 12.1%). Moreover, the grid-testing procedure delivers point estimates of the structural parameters relatively close to the true values. Table 1 also reports the average projected 90% confidence intervals for the individual

[14] To invert the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test numerically, we should consider a multi-dimensional grid search for the log-likelihood $\log L_T(g(\breve{\theta}_s,\theta_\varepsilon))$ on a large number of evenly spaced parameter points. Since in our setup $\dim(\theta_s)$=10 is relatively large, this approach is computationally cumbersome. For instance, if one considers only 10 evenly spaced points within each of the 10 intervals that define the admissibe parameter space (see the last column of Table 1), then it is necessary to evaluate the log-likelihood 10^{10} times for each simulated dataset. To speed up computation time and line with what suggested by Andrews and Mikusheva (2014), we decided to select only 300 points randomly (using the uniform distribution) from the rectangle formed by the Cartesian products of the 10 intervals. Of course, the employment of more sophisticated and efficient algorithms could lead to an even more satisfactorily empirical size-control of the test.

structural parameters (fourth column), and these intervals are contrasted with the actual intervals used to define the parametric grid (fifth column).

5 Empirical evidence

In this section, we apply the '$LR_T \rightarrow AR_T$' testing strategy to post-WWII U.S. monetary policy. We employ quarterly data, sample 1954q3-2008q3, and three observable variables, $X_t{:=}(\tilde{y}_t, \pi_t, R_t)'$. The output gap $\tilde{y}_t$ is computed as percent log-deviation of the real GDP with respect to the potential output estimated by the Congressional Budget Office. The inflation rate π_t is the quarterly growth rate of the GDP deflator. For the short-term nominal interest rate R_t we consider the effective Federal funds rate expressed in quarterly terms (averages of monthly values). The source of the data is the Federal Reserve Bank of St. Louis' web site. The beginning of the sample is due to data availability (in particular, of the effective Federal Funds rate. The end of the sample is justified by our intention to avoid dealing with the 'zero-lower bound' phase began in December 2008, which triggered a series of non-standard policy moves by the Federal Reserve whose effects are hardly captured by our standard new-Keynesian framework.

Our reference structural model is given by the new-Keynesian system (1)-(4). Following most of the literature on the 'Great Moderation', we divide the post-WWII U.S. era in two periods, roughly corresponding to the 'Great Inflation' and the 'Great Moderation' samples. We take the advent of Paul Volcker as Chairman of the Federal Reserve to identify our first sub-sample, i.e. 1954q3-1979q2, which we call 'pre-Volcker' sample. As for the 'Great Moderation' sample, we consider the period 1985q1-2008q3. McConnell and Pérez-Quirós (2000) find a break in the variance of the U.S. output growth in 1984q1. Our empirical investigation deals with a measure of the output gap, inflation, and the federal funds rate. Signs of the 'Volcker disinflation' are still evident in 1984. This is possibly due to the 'credibility build-up' undertaken by the Federal Reserve in the early 1980s, a period during which private agents gradually changed their view on the Federal Reserve's ability to deliver low inflation (Goodfriend and King, 2005). Moreover, the first years of Volcker's tenure (until October 1982) were characterized by non-borrowed reserves targeting. Hence, the fit of our policy rule would substantially worsen if we included the Volcker disinflation (Estrella and Fuhrer, 2003; Mavroeidis, 2010), a fact that would carry consequences on the estimates of all parameters of the system. To circumvent this problem, we postpone the beginning of our second sub-sample to 1985q1. A similar choice is undertaken by Christiano *et al.* (2013). Thus, our 'Great Moderation' sample is given by the period

1985q1-2008q3 and will be denoted as 'post-1985' sample throughout this section.

The first-step of the '$LR_T \rightarrow AR_T$' testing strategy requires computing the 'full-information' $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test discussed in Sub-section 3.1. As is common in the literature, we pre-fix the nominal level of significance at the 10% level (η_1=0.10). The log-likelihood maximization algorithm is inspired to the grid-search approach discussed in Bårdsen and Fanelli (2014). Table 2 summarizes the results of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test on the 'pre-Volcker' and 'post-1985' samples, respectively. In the upper panel of Table 2, we summarize the projected 90% confidence intervals for the individual elements of θ_s derived from the identification-robust confidence set $\mathcal{C}^{*LR}_{0.90}$ (see eq. (21)) and the point estimate of θ_s. The projected confidence intervals are computed using Dufour's (1997) method. In the lower panel, we indicate whether the grid-testing procedure leads to an empty or nonempty identification-robust confidence set, and report the value of $LR_T(\hat{\phi}_{\breve{\theta}_s})$ associated with $\breve{\theta}_{s,ML}$ and corresponding p-value.

Table 2 suggests two important facts. First, the CER that the new-Keynesian system implies under determinacy are firmly rejected on the 'pre-Volcker' sample (the set $\mathcal{C}^{*LR}_{0.90}$ is empty), and are firmly accepted on the 'post-1985' sample by the data (the set $\mathcal{C}^{*LR}_{0.90}$ is nonempty and the p-value associated with the 'least rejected' model is 0.36). We reject the hypothesis of determinacy on the 'pre-Volcker' sample and do not reject the hypothesis H_0' in eq. (15) on the 'post-1985' sample. Despite we can not interpret the result relative to the chosen 'Great Moderation' regime as conclusive evidence of determinacy (see the discussions in Sub-section 2.3 and Sub-section 3.3), our inference is sufficient to rule out the scenario according to which the U.S. business cycle was driven by sunspot expectations extraneous to fundamental shocks. Interestingly, the fact that the CER entailed by the hypothesis of determinacy are not rejected on the period 1985q1-2008q3, suggests an implicit non-rejection of the new-Keynesian system (1)-(4) on that sample. Second, the 90% projected identification-robust confidence intervals for the policy (feedback) parameters $\varphi_{\tilde{y}}$ and φ_π are surprisingly tighter than the confidence sets documented by e.g. Mavroeidis (2010). In particular, the estimation of the value of the parameter φ_π, which captures the systematic reaction of the Federal Reserve to inflation, has attracted a lot of attention. The debate has been intense also because of the lack of precision surrounding the estimates of such parameter. A prominent example in the literature is represented by Mavroeidis (2010). He convincingly shows that, in a single-equation context, the estimation of φ_π tends to be imprecise, and the formal evidence in favor of an aggressive systematic policy response to inflation is scant. Possible reasons include (a) the absence of sunspot shocks under determinacy, which implies a lower volatility of inflation and output and, therefore, a harder identification of the systematic relationship between the policy rate and the policy relevant-macroeconomic variables, and (b) a higher degree of interest rate smoothing, which limits the reaction of the policy rate in presence of shocks hitting inflation and output. Interestingly, our empirical analysis allows us to formally rule out any role for sunspot fluctuations in the 'post-1985' period on the one hand, and a fair amount of interest rate smoothing (ranging from 0.569 to 0.697, according to our 90% confidence interval) by the Federal Reserve, on the other hand. Importantly, our identification-robust approach does not lead us to reject the correct specification of the specified new-Keynesian model during the 'Great Moderation'. Our findings are particularly important in light of a recent paper by Cochrane (2011), who argues that the parameters of Taylor-type rules like that in eq. (3) are not identifiable in prototypical new-Keynesian models. Cochrane (2011), however, considers formulations of the new-Keynesian system which are 'less involved', from a dynamic standpoint, than our 'hybrid' model in eq.s (1)-(4). Table 2 shows that the 'full-information' approach delivers relatively tight confidence sets not only for $\varphi_{\tilde{y}}$ and φ_{π}, but also for δ (intertemporal elasticity of substitution), α (indexation to past inflation), and κ (slope of the NKPC), which are notoriously difficult to estimate precisely from the data.[15]

We then proceed with the 'limited-information' second-step of the '$LR_T \rightarrow tR_T$' testing strategy, which requires the inversion of the Anderson and Rubin-type $AR_T(\breve{\theta}_s)$ test for the OR implied by the system of Euler equations (1)-(4) on the 'pre-Volcker' sample. Recall, indeed, that the CER implied by the new-Keynesian model under the hypothesis of determinacy have been rejected by the data on the 'pre-Volcker' sample. The second-step is conducted to establish whether the rejection of the hypothesis of determinacy can be ascribed to the multiple equilibria hypothesis, or to the inability of the estimated system to capture the propagation mechanisms at work in the data. For completeness, we invert the $AR_T(\breve{\theta}_s)$ test not only on the 'pre-Volcker' sample, but also on the 'post-1985' sample, albeit this calculation would not be required by our testing strategy (recall that we have accepted the new-Keynesian system on the 'post-1985' sample in the previous step). We pre-fix the nominal type-I error η_2 at the level η_2=0.10.

The results of this second-step are summarized in Table 3. In the upper panel, we report the projected confidence intervals for the individual elements

[15]It can be noticed that some of the elements of $\hat{\theta}_{s,ML}$ (fifth column of Table 2) lie exactly on the boundaries of the corresponding intervals used to define the grid (e.g. the point estimate of α). This is perfectly consistent with the identification-robust inference approach, see, e.g., Dufour *et al.* (2006, 2009, 2010, 2013).

of θ_s derived with Dufour's (1997) method from the identification-robust confidence set $\mathcal{C}^{*AR}_{0.90}$, along with the point estimate obtained from the problem in eq. (29) replacing $\mathcal{C}^{AR}_{1-\eta_2}$ with $\mathcal{C}^{*AR}_{0,90}$. In the lower panel, we indicate whether the grid-testing procedure leads to an empty or nonempty identification-robust confidence set and, in the second case, we report the value of the test statistic associated with the point estimate $\hat{\theta}_{s,LI}$ and corresponding p-value.

Table 3 shows that the new-Keynesian model is not rejected by the $AR_T(\breve{\theta}_s)$ test on the 'pre-Volcker' sample (the set $\mathcal{C}^{*AR}_{0.90}$ is nonempty and the p-value associated with the 'least rejected' model is 0.14). As expected, we also find that the new-Keynesian model is not rejected by the $AR_T(\breve{\theta}_s)$ test on the 'post-1985' sample (the set $\mathcal{C}^{*AR}_{0.90}$ is nonempty and the p-value associated with the 'least rejected' model is 0.37). This is a 'reassuring' result, as it corroborates the outcome obtained with the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test in the first-step. Moreover, if we compare the projected identification-robust confidence intervals built with the 'full-information' method (sixth column of Table 2) with the corresponding intervals built with the 'limited-information' method (sixth column of Table 3), we find that the former are remarkably more informative than the latter. This result confirms that 'full-information' methods designed to deal with identification failure provide more precise information than 'limited-information' approaches.

By combining the evidence in Table 3 with that in Table 2, we argue that if one interprets the U.S. business cycle through the lens of the estimated (and not rejected) new-Keynesian system (1)-(4), any inference based on finite-order structural VARs on the 'pre-Volcker' sample is inherently misspecified. Indeed, our test suggests that the 'right' time series model for $X_t:=(\tilde{y}_t, \pi_t, R_t)'$ on the 'pre-Volcker' period belongs to the class of VARMA-type systems in eq.s (11)-(12). Accordingly, any finite-order VAR for X_t would represent a truncated approximation to the actual equilibrium and might in principle return largely incorrect estimates of the impulse response function and the parameters of interest; see e.g. Ravenna (2007) for a similar point.

Overall, we can conclude that the '$LR_T \rightarrow AR_T$' testing strategy leads us to accept the hypothesis of indeterminacy (H_1' in eq. (16)) on the 'pre-Volcker' sample, for which the set $\mathcal{C}^{*LR}_{0.90}$ is empty and the set $\mathcal{C}^{*AR}_{0.90}$ is nonempty, and not to reject the hypothesis H_0' in eq. (15) on the 'Great Moderation' sample, for which the set $\mathcal{C}^{*LR}_{0.90}$ is nonempty. Our conclusions are consistent with the occurrence of a policy switch in the late 1970s. Our prior-free approach maximizes the role attached to the data in determining these results.[16]

[16] An approximate and purely indicative measure of the extent of the change characterizing the parameters of the model across the two regimes can be broadly obtained by comparing the identification-robust confidence intervals and the point estimates reported

6 Relation to the literature

Our paper has several connections with the literature. On the methodological side, our analysis is related to the recent works of Guerron-Quintana *et al.* (2013) and Dufour *et al.* (2013) on identification-robust frequentist inference in DSGE models. The first-step of our testing procedure is essentially based on the pointwise inversion of the likelihood ratio test proposed by Guerron-Quintana *et al.* (2013) as a tool to build identification-robust confidence sets for the structural parameters. Our methodology is also connected to the contributions by Stock and Wright (2000), Kleiberger and Mavroeidis (2009) and Dufour *et al.* (2006, 2009, 2010, 2013), among others. Indeed, conditional on the first-step, the second-step of the suggested testing strategy requires the pointwise inversion of an Anderson Rubin-type test for the OR implied by the system of Euler equations. Compared to Fanelli (2012), who proposes a test for determinacy/indeterminacy in new-Keynesian models controlling for the omission of propagation mechanisms, our procedure is robust to identification failures and can be applied regardless of the strength of identification. Moreover, the logic of the test and its properties are completely different: while we test the OR in the system of Euler equations only if the CER obtained under determinacy are rejected in the first-step, in Fanelli (2012) the CER obtained under determinacy are tested in a second-step, conditionally on the OR implied by the system of Euler equations not being rejected in the first-step.

Finally, it worth stressing that our testing approach is not related to situations in which the agents know that an economy fluctuates between determinate and indeterminate states driven by a Markov-switching process as in e.g. Farmer *et al.* (2009).

On the empirical side, Lubik and Schorfheide (2004) test for determinacy in the U.S. economy with a model similar to ours, by undertaking a Bayesian investigation in which posterior weights for the determinacy and indeterminacy regions of the parameter space are constructed and compared. Our paper implements a frequentist approach, which neither requires the use of a-prior distributional assumptions, nor the commitment to non-standard in-

in Table 2 and Table 3. For instance, we find that as for the parameters δ (intertemporal elasticity of substitution) α (indexation to past inflation), φ_π (long run reaction to inflation) and ρ_π (inflation shock persistence), the 'full-information' point estimates computed on the 'post-1985' sample (see the fifth column of Table 2) do not lie within (or lie on the border of) the corresponding 'limited-information' identification-robust confidence intervals computed on the 'pre-Volcker' sample (see the fourth column of Table 3). Evidence of instability in the parameters of the private sector, other than the policy parameters, has also been found, among others, by Canova (2009), Inoue and Rossi (2011*a*), Canova and Menz (2011), Canova and Ferroni (2012), Castelnuovo (2012a), and Cantore *et al.* (2013).

ference. In particular, we are not forced to choose a prior distribution for some arbitrary auxiliary parameters that index the multiplicity of solutions under rational expectations as in Lubik and Schorfheide (2004). With respect to Boivin and Giannoni (2006), our method is based on the direct estimation of the structural new-Keynesian model and provides a direct control for the cases of identification failure and dynamic misspecification. Hence, we need not minimize the distance between some selected impulse responses taken from a VAR modeling the macroeconomic variables of interest and the structural model-based responses, a methodology which is unfortunately bias-prone as for expectations-based models like ours (Canova and Sala, 2009). More importantly, we need not make restrictive assumptions on the solution under indeterminacy, as opposed to the MSV solution assumed by Boivin and Giannoni (2006). While being plausible, such solution is anyhow arbitrary, and it may importantly affect the simulated moments of interest (Castelnuovo, 2012*b*).

Mavroeidis (2010) applies identification-robust 'limited-information' methods to investigate the determinacy/indeterminacy of U.S. monetary policy conditional on the estimation of the policy rule in isolation. Compared to Mavroeidis (2010), we investigate the issue of macroeconomic stability of U.S. monetary policy by using a fully specified 'hybrid new-Keynesian model' à la Benati and Surico (2009), and apply a testing strategy which combines 'limited-' and 'full-information' methods and is robust to identification failure. Mavroeidis (2010) conjectures that the difference between the (precise) confidence intervals in the 'pre-Volcker' period and the (imprecise) ones in the 'post-Volcker' phase may be interpreted as (a) absence of sunspot fluctuations during the 'Great Moderation'; (b) increase in the policy inertia; (c) larger variability of the policy shocks during the first years of the Volcker era. Our methodology formally shows that sunspot fluctuations are unlikely to have played a role during the 'Great Moderation'. We therefore offer statistical support to Mavroeidis' conjecture (a). Differently, we do not find clear evidence in favor of an increase in the policy inertia when moving from our first to our second sub-sample. However, the confidence interval surrounding the point estimate of the degree of interest rate smoothing during the 'Great Moderation' does not exclude Mavroeidis' second conjecture (b) either. Finally, our 'Great Moderation' sub-sample begins in 1985, i.e., after the end of the 'Volcker experiment' related to the targeting of non-borrowed reserves by the Federal Reserve. Hence, our results are not necessarily driven by a large volatility of the policy shocks, whose variance has drastically reduced since 1985 (see Mavroeidis (2010), Figure 3 - left panel). More importantly, however, we show that, when applying a system based 'full-information' approach designed to handle weak identification, the precision of the estimates obtained for the 'Great Moderation' sample is higher than the one achieved via a single-equation approach.

7 Concluding remarks

This paper has proposed and implemented a novel identification-robust approach to test the null hypothesis that a fully specified small-scale new-Keynesian monetary policy model has a reduced form consistent with the unique stable solution, versus the alternative of indeterminacy. The testing strategy is designed such that when the null hypothesis is rejected, a second-step is run to establish whether the rejection is due to the occurrence of multiple equilibria or to the omission of relevant propagation mechanisms from the specified system of structural Euler equations. Our methodology can be applied regardless of the strength of identification of the structural parameters, and it requires neither the use of prior distributions nor that of nonstandard inference. Hence, our procedure works in favor of reducing the degree of arbitrariness of our empirical results.

We have applied our novel methodology to a standard dataset of U.S. macroeconomic data by using the new-Keynesian framework recently employed by Benati and Surico (2009) as our reference structural model. The results of our testing strategy conform to the case of a switch from indeterminacy to a framework consistent with determinacy, in correspondence to the advent of Paul Volcker as Chairman of the Federal Reserve. Nevertheless, it is not possible to claim that our analysis supports the hypothesis of a unique equilibrium after Volcker. With respect to Mavroeidis (2010), who works with a single-equation 'limited-information' approach, we find tighter confidence bands for our estimated parameters. We attribute this difference to the 'full-information' nature of the first-step of our robust test and to the fact that the estimated new-Keynesian model is not rejected by the data on the 'Great Moderation' period.

To be clear, our findings, which line up with a number of previous contributions in the literature, are consistent with, but do not necessarily point to, the 'good policy' explanation of the U.S. Great Moderation. In light of the recent financial crisis, our analysis as for the period mid-1980s-onwards may very well be over. When enough data become available, our methodology will help to shed further light on this issue.

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TABLES

Table 1. Empirical size of the '$LR_T \rightarrow AR_T$' testing strategy when the data are generated from the new-Keynesian system (5)-(6) under the hypothesis H_0' in eq. (15).

'true' $\theta_{0,s}$		T=100	η_1=0.10	
$\lambda_{\max}(G(\theta_{0,s}))$:=0.964	Interpret.	$\hat{\theta}_{s,ML}$	Avg. proj. 90% c.i. & grid int.	
γ_0:=0.744	IS, forw. look. term	0.694 (0.206)	[0.728-0.784]	[0.688-0.822]
δ_0:=0.124	IS, inter. elast. of subst.	0.117 (0.038)	[0.113-0.141]	[0.090-0.160]
α_0:=0.059	NKPC: index, past infl.	0.058 (0.026)	[0.047-0.081]	[0.030-0.099]
κ_0:=0.044	NKPC: slope	0.041 (0.013)	[0.039-0.051]	[0.035-0.056]
ρ_0:=0.834	Rule, smoothing term	0.747 (0.224)	[0.772-0.841]	[0.515-0.877]
$\varphi_{\widetilde{y},0}$:=1.146	Rule, react. to out. gap	0.925 (0.434)	[0.705-1.237]	[0.383-1.610]
$\varphi_{\pi,0}$:=1.749	Rule, reaction to infl.	1.463 (0.637)	[1.228-1.917]	[0.700-2.570]
$\rho_{\widetilde{y},0}$:=0.796	Out. gap shock, pers.	0.729 (0.215)	[0.765-0.818]	[0.738-0.834]
$\rho_{\pi,0}$:=0.418	Infl. shock, pers.	0.378 (0.126)	[0.356-0.462]	[0.300-0.520]
$\rho_{R,0}$:=0.404	Pol. rate shock, pers.	0.371 (0.125)	[0.354-0.453]	[0.289-0.518]
		Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.079	Rej($LR_T(\hat{\phi}_{\theta_{s,0}})$)=0.121	

NOTES. Results are obtained using M=1,000 replications. Each simulated sample is initiated with 200 additional observations to get a stochastic initial state and then are discarded. The structural parameters are calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico (2009), column 'After the Volcker stabilization'. The numerical inversion of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test for the CER (first-step) is obtained on each generated dataset by considering 300 points $\breve{\theta}_s$ randomly chosen (using the uniform distribution) from the grid delimited by the space formed by the Cartesian product of the intervals reported in the last column. '$\hat{\theta}_{s,ML}$' is the vector of point estimates of θ_s obtained from the problem in eq. (22) by replacing $\mathcal{C}^{LR}_{1-\eta_1}$ with $\mathcal{C}^{*LR}_{0.90}$ (corresponding Monte Carlo standard errors in brackets). 'Average proj. 90% c.i. & grid intervals' reports the average projected 90% confidence interval computed as in Dufour (1997) and the actual intervals used for the individual structural parameters in the grid testing procedure. 'Rej(·)' stands for 'rejection frequency'. $LR_T(\hat{\phi}_{\theta_{s,0}})$ is the test statistic for the hypothesis $H_{0,cer}$ in eq. (20) evaluated at the specific point $\breve{\theta}_s = \theta_{s,0}$.

Table 2. Projected 90% identification-robust confidence intervals, point estimates of the structural parameters θ_s:=$(\gamma, \delta, \alpha, \kappa, \rho, \varphi_{\tilde{y}}, \varphi_\pi, \rho_{\tilde{y}}, \rho_\pi, \rho_R)'$ and results of the first-step of the '$LR_T \rightarrow AR_T$' testing strategy on U.S. quarterly data.

		1954q3-1979q2 'pre-Volcker'		1985q1-2008q3 'Gr. Moder.'	
Param.	Interpretation	$\hat{\theta}_{s,ML}$	proj. 90% c.i.	$\hat{\theta}_{s,ML}$	proj. 90% c.i.
γ	IS, forw. look. term	-	-	0.729	[0.652-0.772]
δ	IS, inter. elast. of subst.	-	-	0.082	[0.082-0.154]
α	NKPC: index, past infl.	-	-	0.020	[0.020-0.059]
κ	NKPC: slope	-	-	0.048	[0.042-0.098]
ρ	Rule, smoothing term	-	-	0.666	[0.569-0.697]
$\varphi_{\tilde{y}}$	Rule, react. to out. gap	-	-	0.339	[0.127-0.479]
φ_π	Rule, reaction to infl.	-	-	5.439	[2.318-5.445]
$\rho_{\tilde{y}}$	Out. gap shock, pers.	-	-	0.920	[0.720-0.978]
ρ_π	Infl. shock, pers.	-	-	0.925	[0.748-0.970]
ρ_R	Pol. rate shock, pers.	-	-	0.794	[0.730-0.806]
identification-robust c.s. $\mathcal{C}_{0.90}^{*LR}$		empty		nonempty (card($\mathcal{C}_{0.90}^{*LR}$)=15)	
$\lambda_{\max}(G(\hat{\theta}_{s,ML}))$		–		0.946	
$LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ test (first-step)		–		19.54 [0.36]	

NOTES. The projected 90% identification-robust confidence intervals (proj. 90% c.i.) have been obtained from the 90% identification-robust confidence set $\mathcal{C}_{0.90}^{*LR}$ (see eq. (30)) as in Dufour (1997). The set $\mathcal{C}_{0.90}^{*LR}$ has been obtained by inverting numerically the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test considering 5,000,000 points $\breve{\theta}_s$ chosen randomly (using the uniform distribution) from the rectangle formed by the Cartesian product of the following intervals: [0.65, 0.85] for γ, [0.08, 0.16] for δ, [0.02, 0.10] for α, [0.04, 0.10] for κ, [0.50, 0.70] for ρ, [0.05, 1.5] for $\varphi_{\tilde{y}}$, [0.5, 5.5] for φ_π, [0.40, 0.98] for $\rho_{\tilde{y}}, \rho_\pi$ and ρ_R. '$\hat{\theta}_{s,ML}$' is the point estimate derived from the problem in eq. (22) replacing $\mathcal{C}_{1-\eta_1}^{LR}$ with $\mathcal{C}_{0.90}^{*LR}$. $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ correspondes to the value of the test statistics obtained in correspondence of the 'least rejected' model within $\mathcal{C}_{0.90}^{*LR}$. P-values in brackets. Estimation on each sub-period is carried out by considering within-periods initial values and variables are demeaned within each sub-period.

Table 3. Projected 90% identification-robust confidence intervals, point estimates of the structural parameters θ_s:=$(\gamma, \delta, \alpha, \kappa, \rho, \varphi_{\tilde{y}}, \varphi_\pi, \rho_{\tilde{y}}, \rho_\pi, \rho_R)'$ and results of the second-step of the '$LR_T \rightarrow AR_T$' procedure on U.S. quarterly data.

		1954q3-1979q2 'pre-Volcker'		1985q1-2008q3 'Gr. Moder.'	
Param.	Interpretation	$\hat{\theta}_{s,LI}$	proj. 90% c.i.	$\hat{\theta}_{s,LI}$	proj. 90% c.i.
γ	IS, forw. look. term	0.841	[0.660-0.845]	0.821	[0.650-0.850]
δ	IS, inter. elast. of subst.	0.088	[0.084-0.160]	0.132	[0.080-0.160]
α	NKPC: index, past infl.	0.025	[0.020-0.070]	0.097	[0.020-0.099]
κ	NKPC: slope	0.042	[0.040-0.058]	0.087	[0.040-0.100]
ρ	Rule, smoothing term	0.520	[0.500-0.698]	0.699	[0.500-0.700]
$\varphi_{\tilde{y}}$	Rule, react. to out. gap	0.138	[0.050-0.325]	0.295	[0.050-1.043]
φ_π	Rule, reaction to infl.	0.687	[0.500-0.906]	2.123	[0.500-5.499]
$\rho_{\tilde{y}}$	Out. gap shock, pers.	0.900	[0.620-0.964]	0.911	[0.400-0.980]
ρ_π	Infl. shock, pers.	0.578	[0.414-0.793]	0.907	[0.400-0.980]
ρ_R	Pol. rate shock, pers.	0.798	[0.565-0.916]	0.795	[0.674-0.980]
identification-robust c.s. $\mathcal{C}^{*AR}_{0.90}$		nonempty (card($\mathcal{C}^{*AR}_{0.90}$)=26)		nonempty (card($\mathcal{C}^{*AR}_{0.90}$)=41891)	
$\lambda_{\max}(G(\hat{\theta}_{s,LI}))$		1.012		0.965	
$AR_T(\hat{\theta}_{s,LI})$ test (second-step)		24.44 [0.14]		19.27 [0.37]	

NOTES. The projected 90% identification-robust confidence intervals (proj. 90% c.i.) have been obtained from the 90% identification-robust confidence set $\mathcal{C}^{*AR}_{0.90}$ (see eq. (31)) as Dufour (1997). The confidence sets have been obtained by inverting the test $AR_T(\breve{\theta}_s)$ (second-step); in practice, $AR_T(\breve{\theta}_s)$ is computed as a quasi-LR test using Z_t:=$(X'_{t-1}, X'_{t-2})'$ in the auxiliary multivariate regression system (26), considering 5,000,000 points $\breve{\theta}_s$ randomly chosen (using the uniform distribution) from the rectangle formed by the Cartesian product of the same intervals as in Table 2 and imposing the condition $\lambda_{\max}(G(\breve{\theta}_s))$>1 on the 1954q3-1979q2 period. '$\hat{\theta}_{s,LI}$' is the point estimate derived the problem in eq. (29) by replacing $\mathcal{C}^{AR}_{1-\eta_2}$ with $\mathcal{C}^{*AR}_{0.90}$. $AR_T(\hat{\theta}_{s,LI})$ reports the value of the test statistics obtained in correspondence of the 'least rejected' model within $\mathcal{C}^{*AR}_{0.90}$. P-values in brackets. Estimation on each sub-period is carried out by considering within-periods initial values and variables are demeaned within each sub-period.

Technical Supplement of "Monetary Policy Indeterminacy and Identification Failures in the U.S.: Results from a Robust Test" by Efrem Castelnuovo and Luca Fanelli

Introduction

In this supplementary material, we (i) specify all assumptions underlying the new-Keynesian system analyzed in the paper, (ii) derive its reduced form solutions, (iii) formalize some asymptotic properties of the '$LR_T \to AR_T$' testing strategy and (iv) complete the Monte Carlo experiment presented in the paper with further results.

Structural model, assumptions and time series representations

The structural model is given by

$$\Gamma_0 X_t = \Gamma_f E_t X_{t+1} + \Gamma_b X_{t-1} + \omega_t \tag{32}$$

$$\omega_t = \Xi \omega_{t-1} + \varepsilon_t \ , \ \ \varepsilon_t \sim \text{WN}(0, \Sigma_\varepsilon) \tag{33}$$

where the matrices Γ_0, Γ_f, Γ_b, Ξ and Σ_ε depend nonlinearly on the $m \times 1$ vector of structural parameters θ, X_t is the vector of modeled variables, ω_t stands for the vector of autoregressive stochastic processes hitting the system, and ε_t is the vector of orthogonal martingale differences which we interpret as structural shocks. The space of all theoretically admissible values of θ is denoted by $\mathcal{P}$ and is assumed to be compact. Expectations are conditional on the information set $\mathcal{F}_t$, i.e. $E_t\cdot{:=}E(\cdot \mid \mathcal{F}_t)$. We consider the partition $\theta{:=}(\theta_s', \ \theta_\varepsilon')'$, where the sub-vector θ_ε contains the non-repeated, non-zero elements of $vech(\Sigma_\varepsilon)$. Given the partition $\theta{:=}(\theta_s', \ \theta_\varepsilon')'$, we also consider the corresponding partition of the parameter space $\mathcal{P}{:=}\mathcal{P}_{\theta_s} \times \mathcal{P}_{\theta_\varepsilon}$.The true value of θ, $\theta_0{:=}(\theta_{0,s}', \ \theta_{0,\varepsilon}')'$, is an interior point of $\mathcal{P}$.

Throughout the paper it will be maintained that $\dim(X_t) = \dim(\varepsilon_t){:=}n > 1$. Moreover, we use the notations '$A(\theta)$' and '$A{:=}A(\theta)$' interchangeably to indicate that the elements of the matrix A depend nonlinearly on the structural parameters θ. In our setup, $\Gamma_0{:=}\Gamma_0(\theta_s)$, $\Gamma_f{:=}\Gamma_f(\theta_s)$, $\Gamma_b{:=}\Gamma_b(\theta_s)$, $\Xi{:=}\Xi(\theta_s)$ and $\Sigma_\varepsilon{:=}\Sigma_\varepsilon(\theta_\varepsilon)$.

We consider the following assumptions.

Assumption 1 The matrix $\Gamma_0^{\Xi}{:=}(\Gamma_0+\Xi\Gamma_f)$ is non-singular and Ξ is stable; the matrix $(\Gamma_0^{\Xi}-\Gamma_f\Phi_{c,1})$ is non-singular, where $\Phi_{c,1}{:=}\Phi_{c,1}(\theta)$ is a 3×3 matrix.

Assumption 2 For $\theta\in\mathcal{P}$, any reduced form solution to system (32)-(33) is covariance stationary.

Using some algebra, system (32)-(33) can written in the form

$$\Gamma_0^{\Xi}X_t=\Gamma_f E_t X_{t+1}+\Gamma_{b,1}^{\Xi}X_{t-1}+\Gamma_{b,2}^{\Xi}X_{t-2}+\varepsilon_t^{\Xi} \tag{34}$$

$$\begin{aligned}
&\Gamma_0^{\Xi}{:=}(\Gamma_0+\Xi\Gamma_f)\\
&\Gamma_{b,1}^{\Xi}{:=}(\Gamma_b+\Xi\Gamma_0)\\
&\Gamma_{b,2}^{\Xi}{:=}-\Xi\Gamma_b
\end{aligned}$$

where the 'composite' structural disturbance $\varepsilon_t^{\Xi}{:=}\varepsilon_t+\Xi\Gamma_f\eta_t$, $\eta_t{:=}(X_t-E_{t-1}X_t)$ is a Martingale Difference Sequence (MDS) with respect to $\mathcal{F}_t$, because of the MDS property of η_t.

We rewrite system (34) in canonical form (Binder and Pesaran, 1995). To do this, define the $2n\times 1$ state vectors $\mathring{X}_t{:=}(X_t',X_{t-1}')'$ and $\mathring{\varepsilon}_t{:=}(\varepsilon_t^{\Xi\prime},0_{n\times 1}')'$, and then express the system in the form

$$\mathring{\Gamma}_0\mathring{X}_t=\mathring{\Gamma}_f E_t\mathring{X}_{t+1}+\mathring{\Gamma}_b\mathring{X}_{t-1}+\mathring{\varepsilon}_t \tag{35}$$

where

$$\mathring{\Gamma}_0{:=}\begin{pmatrix}\Gamma_0^{\Xi} & 0_{n\times n}\\ 0_{n\times n} & I_n\end{pmatrix}\quad,\quad \mathring{\Gamma}_f{:=}\begin{pmatrix}\Gamma_f & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n}\end{pmatrix}\ ,\ \mathring{\Gamma}_b{:=}\begin{pmatrix}\Gamma_{b,1}^{\Xi} & \Gamma_{b,2}^{\Xi}\\ I_n & 0_{n\times n}\end{pmatrix}.$$

By inverting $\mathring{\Gamma}_0$ (Assumption 1) in system (35) we obtain

$$\mathring{X}_t=\mathring{D}E_t\mathring{X}_{t+1}+\mathring{B}\mathring{X}_{t-1}+\mathring{w}_t \tag{36}$$

where $\mathring{D}{:=}\mathring{\Gamma}_0^{-1}\mathring{\Gamma}_f$, $\mathring{B}{:=}\mathring{\Gamma}_0^{-1}\mathring{\Gamma}_b$ and $\mathring{w}_t{:=}\mathring{\Gamma}_0^{-1}\mathring{\varepsilon}_t$. In general, the matrices $\mathring{D}$ and $\mathring{B}$ can be singular.

A solution to system (36) is any process $\left\{\mathring{X}_t^*\right\}_{t=0}^{\infty}$ such that *(a)* the quantity $E_t\mathring{X}_{t+1}^*$ exists and *(b)* when $\mathring{X}_t{:=}\mathring{X}_t^*$ is substituted into the model, the equations of the system are verified at any time t for given initial conditions $\mathring{X}_0{:=}(X_0',X_{-1}')'$. If $\left\{\mathring{X}_t^*\right\}_{t=0}^{\infty}$ is a solution of system (36), then $X_t{:=}H\mathring{X}_t^*$, where $H{:=}[I_n\ ,\ 0_{n\times n}]$, is a known selection matrix, will be a solution to system (34). We define a reduced form solution of system (32)-(33) any member of the solution set such that X_t can be expressed as linear function of ε_t, lags of X_t and ε_t, and possibly other components which are function of MDSs with respect $\mathcal{F}_t$. The reduced form solution is 'stable' if the companion matrix associated with the companion form representation of the reduced form solution is stable. We call stable a matrix that has all eigenvalues inside the unit disk, and 'unstable' a matrix that has at least one eigenvalue outside the unit disk. Thus, denoting with $\lambda_{\max}(\cdot)$ the absolute value of the largest eigenvalue of the matrix in the argument, the condition $\lambda_{\max}(A(\theta))<1$ holds for stable matrices, and $\lambda_{\max}(A(\theta))>1$ for unstable ones. The reduced form solution is 'unique' if it time series representation involves only X_t and ε_t, and the conditional distribution of X_t given $\mathcal{F}_{t-1}$ depends only on θ.

Before presenting our main results, we sketch some features of the solution method used in the paper. Following Binder and Pesaran (1995), given a solution $\left\{\mathring{X}_t^*\right\}_{t=0}^{\infty}$, we assume that $\mathring{X}_t{:=}\mathring{X}_t^*$ is decomposed into two components, i.e.

$$\mathring{X}_t{:=}\mathring{X}_{B,t}+\mathring{X}_{F,t} \tag{37}$$

$$\mathring{X}_{B,t}{:=}\mathring{\Phi}\mathring{X}_{t-1} \tag{38}$$

where the process $\left\{\mathring{X}_{B,t}\right\}_{t=0}^{\infty}$ represents the 'backward' part of the solution, and the process $\left\{\mathring{X}_{F,t}\right\}_{t=0}^{\infty}$ is its 'forward' part. In particular, $\mathring{X}_{F,t}$ is assumed to be a solution to the 'Cagan multivariate' model

$$\mathring{X}_{F,t}=\mathring{C}_f E_t\mathring{X}_{F,t+1}+\mathring{C}_0\mathring{w}_t \tag{39}$$

for given choice of the matrices $\mathring{C}_f{:=}\mathring{C}_f(\theta)$ and $\mathring{C}_0{:=}\mathring{C}_0(\theta)$.

Eq.(38) posits that $\mathring{X}_{B,t}$ obeys an autoregressive scheme. The $2n\times 2n$ matrix $\mathring{\Phi}$ must be real and stable under Assumption 2. Assumption 2 also ensures that only non-explosive stable solutions of system (39) will be considered. From eq.(38) it turns out that

$$E_t\mathring{X}_{F,t+1}=E_t\mathring{X}_{t+1}-\mathring{\Phi}\mathring{X}_t \tag{40}$$

so using eq.(40) in eq.(36), yields

$$\mathring{X}_{B,t}+\mathring{X}_{F,t}=\mathring{D}[E_t\mathring{X}_{F,t+1}+\mathring{\Phi}\mathring{X}_t]+\mathring{B}\mathring{X}_{t-1}+\mathring{w}_t$$

and this system can be re-arranged in the form

$$(I_{2n}-\mathring{D}\mathring{\Phi})\mathring{X}_{F,t}=\mathring{D}E_t\mathring{X}_{F,t+1}+(\mathring{D}\mathring{\Phi}^2-\mathring{\Phi}+\mathring{B})\mathring{X}_{t-1}+\mathring{w}_t. \tag{41}$$

We observe that if there exists a stable matrix $\mathring{\Phi}$, denoted with $\mathring{\Phi}_c$, which satisfies the restriction

$$\mathring{D}\mathring{\Phi}_c^2 - \mathring{\Phi}_c + \mathring{B} = 0_{2n\times 2n}, \tag{42}$$

then the system of equations (41) collapses to

$$(I_{2n} - \mathring{D}\mathring{\Phi}_c)\mathring{X}_{F,t} = \mathring{D}E_t\mathring{X}_{F,t+1} + \mathring{w}_t \tag{43}$$

and can be expressed in the 'Cagan multivariate' form in eq.(39) if the matrix $(I_{2n} - \mathring{D}\mathring{\Phi}_c)$ is invertible.

Inspection of the block structure of the matrices in eq.(42) shows that the form of the matrix $\mathring{\Phi}_c$ is given by

$$\mathring{\Phi}_c = \begin{pmatrix} \Phi_{c,1} & \Phi_{c,2} \\ I_n & 0_{n\times n} \end{pmatrix} \tag{44}$$

where $\Phi_{c,1}:=\Phi_{c,1}(\theta)$ and $\Phi_{c,2}:=\Phi_{c,2}(\theta)$ depend on θ, while

$$(I_{2n} - \mathring{D}\mathring{\Phi}_c) = \begin{pmatrix} I_n - \left(\Gamma_0^{\Xi}\right)^{-1}\Gamma_f\Phi_{c,1} & -\left(\Gamma_0^{\Xi}\right)^{-1}\Gamma_f\Phi_{c,2} \\ 0_{n\times n} & I_n \end{pmatrix}.$$

This matrix is non-singular by Assumption 1, hence we can re-write the system (43) in the multivariate Cagan form in eq.(39) based on

$$\mathring{C}_f := (I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1}\mathring{D} = \begin{pmatrix} G(\theta) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{45}$$

$$\mathring{C}_0 := (I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1} \tag{46}$$

where $G(\theta):=(\Gamma_0^{\Xi} - \Gamma_f\Phi_{c,1})^{-1}\Gamma_f$. It turns out that the stability/instability of the matrix $G(\theta)$ determines the stability/instability of $\mathring{C}_f$ and therefore the solution properties of the system. Note that $G(\theta) = G(\theta_s)$, i.e. the $G(\cdot)$ matrix does not depend on the parameters θ_ε associated with the covariance matrix of the structural shocks.

We can now prove our main results. We report below two propositions and one corollary. Proposition 1 posits that for a given $\theta_s{=}\breve{\theta}_s$, the condition $\lambda_{\max}(G(\breve{\theta}_s)){<}1$ is sufficient for the existence of the finite-order VAR solution for X_t discussed in Sub-section 2.2 of the paper. Proposition 2 establishes that for a given $\theta_s{=}\breve{\theta}_s$, the condition $\lambda_{\max}(G(\breve{\theta}_s)){>}1$ is sufficient for the existence of the VARMA-type indeterminate reduced form solutions for X_t discussed in Sub-section 2.3 of the paper. Finally, Corollary 1 proves that the inequality $\lambda_{\max}(G(\breve{\theta}_s)){>}1$ is also necessary for the existence of the class of VARMA-type reduced form solutions.

Proposition 1 [Sufficient condition for the finite-order VAR reduced form solution] Consider the new-Keynesian system (32)-(33) and Assumptions 1-2. If for $\theta_s{=}\breve{\theta}_s$, $\lambda_{\max}(G(\breve{\theta}_s)){<}1$, the reduced form solution is stable and can be represented as the finite-order VAR

$$(I_n - \Phi_{c,1}(\breve{\theta}_s)L - \Phi_{c,2}(\breve{\theta}_s)L^2)X_t = \Upsilon(\breve{\theta}_s)^{-1}\varepsilon_t \tag{47}$$

where L is the lag/lead operator ($L^h X_t{:=}X_{t-h}$), X_0 and X_{-1} are fixed initial conditions, $\Phi_{c,1}(\breve{\theta}_s)$ and $\Phi_{c,2}(\breve{\theta}_s)$ are sub-matrices of the stable matrix $\mathring{\Phi}_c$ in eq. (44) which solves the quadratic matrix equation

$$\mathring{\Gamma}_f\mathring{\Phi}_c^2 - \mathring{\Gamma}_0\mathring{\Phi}_c + \mathring{\Gamma}_b = 0_{2n\times 2n}, \tag{48}$$

and $\Upsilon(\breve{\theta}_s){:=}(\Gamma_0 - \Gamma_f\Phi_1(\breve{\theta}_s))$. The solution in eq.(47) is also unique according to our definition.

Proof (heuristic). The condition $\lambda_{\max}(G(\breve{\theta}_s)){<}1$ implies $\lambda_{\max}(\mathring{C}_f){<}1$ which in turn implies that $\mathring{C}_f$ is absolutely summable. Under this condition, the unique stable ('bubbles-free') solution of system (39) based on $\mathring{C}_f{:=}(I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1}\mathring{D}$ and $\mathring{C}_0{:=}(I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1}$, is given by

$$\mathring{X}_{F,t}{=}\sum_{j=0}^{\infty}\left(\mathring{C}_f\right)^j\mathring{C}_0E_t\mathring{w}_{t+j}{=}\mathring{C}_0\mathring{w}_t{=}(I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1}\mathring{w}_t.$$

Using eq.s (37)-(38) and $\mathring{w}_t{:=}\mathring{\Gamma}_0^{-1}\mathring{\varepsilon}_t$, one obtains

$$\mathring{X}_t{=}\mathring{\Phi}_c(\theta)\mathring{X}_{t-1} + (\mathring{\Gamma}_0 - \mathring{\Gamma}_f\mathring{\Phi}_c)^{-1}\mathring{\varepsilon}_t.$$

In matrix form, this system reads

$$\begin{pmatrix} X_t \\ X_{t-1}\end{pmatrix} = \begin{pmatrix} \Phi_{c,1}(\theta) & \Phi_{c,2}(\theta) \\ I_n & 0_{n\times n}\end{pmatrix}\begin{pmatrix} X_{t-1} \\ X_{t-2}\end{pmatrix} + \begin{pmatrix} \Theta & -\Gamma_f\Phi_{c,2} \\ 0_{n\times n} & I_n\end{pmatrix}^{-1}\begin{pmatrix} \varepsilon_t^{\Xi} \\ 0_{n\times 1}\end{pmatrix}$$

where $\Theta{:=}(\Gamma_0^{\Xi} - \Gamma_f\Phi_{c,1})$, $\Phi_{c,1} = \Phi_{c,1}(\theta)$, and $\Phi_{c,1}(\theta)$ and $\Phi_{c,2}(\theta)$ are obtained from the matrix $\mathring{\Phi}_c$ in eq. (44), i.e. the stable matrix that solves eq. (48). Considering the first block of n equations of this system, we infer that the reduced form solution must be such that $\eta_t{:=}X_t - E_{t-1}X_t{=}\Theta^{-1}\varepsilon_t^{\Xi}$. Using the definition $\varepsilon_t^{\Xi}{:=}\varepsilon_t + \Xi\Gamma_f\eta_t$ (see system (35)), we obtain the relationship $\Theta\eta_t{=}(\varepsilon_t{+}\Xi\Gamma_f\eta_t)$ which, solved for ε_t, gives:

$$\varepsilon_t{=}\Theta\eta_t - \Xi\Gamma_f\eta_t{=}(\Theta - \Xi\Gamma_f)\eta_t{=}(\Gamma_0^{\Xi} - \Gamma_f\Phi_{c,1} - \Xi\Gamma_f)\eta_t$$

$$=(\Gamma_0 + \Xi\Gamma_f - \Gamma_f\Phi_{c,1} - \Xi\Gamma_f)\eta_t = (\Gamma_0 - \Gamma_f\Phi_{c,1})\eta_t{=}\Upsilon\eta_t.$$

In light of this relationship, the quantity $\eta_t{:=}X_t - E_{t-1}X_t$ is equivalent to

$$X_t - E_{t-1}X_t = \Upsilon(\breve{\theta}_s)^{-1}\varepsilon_t \quad , \quad \Upsilon(\breve{\theta}_s){:=}(\Gamma_0 - \Gamma_f\Phi_{c,1}).$$

Using the lag operator, the equation above is equivalent to system (47). This solution does not involve extra parameters other than θ and extra shock terms other than ε_t, hence it is unique according to our definition.■

Proposition 2 [Sufficient condition for the VARMA-type ind. reduced form solution Consider the new-Keynesian system (32)-(33) and Assumptions 1-2. If for $\theta_s{=}\breve{\theta}_s$, $\lambda_{\max}(G(\breve{\theta}_s)){>}1$ and all eigenvalues of the matrix $\Xi(\breve{\theta}_s)\Gamma_f(\breve{\theta}_s)M(\breve{\theta}_s,\psi)\Theta(\breve{\theta}_s)^{-1}$ defined below are different from 1, there are multiple stable solutions which can be represented in the form:

$$(I_3{-}\Pi(\breve{\theta}_s)L)(I_3{-}\Phi_{c,1}(\breve{\theta}_s)L{-}\Phi_{c,2}(\breve{\theta}_s)L^2)X_t = (M(\breve{\theta}_s,\psi){-}\Pi(\breve{\theta}_s)L)V(\breve{\theta}_s,\psi)^{-1}\varepsilon_t{+}\tau_t \tag{49}$$

$$\tau_t{:=}(M(\breve{\theta}_s,\psi) - \Pi(\breve{\theta}_s)L)V(\breve{\theta}_s,\psi)^{-1}P(\breve{\theta}_s)\zeta_t + P(\breve{\theta}_s)\zeta_t. \tag{50}$$

In eq.s (49)-(50), L is the lag/lead operator ($L^hX_t{:=}X_{t-h}$), X_0, X_{-1} and X_{-2} are fixed initial conditions; $\zeta_t{:=}(0'_{n_1\times 1}, s'_t)'$ and s_t is a $n_2 \times 1$ vector ($n_2{:=}n - n_1$, $n_2 \leq n$) of MDS called sunspot shocks; $\Phi_{c,1}(\breve{\theta}_s)$ and $\Phi_{c,2}(\breve{\theta}_s)$ are sub-matrices of the stable matrix Φ_c which solves eq.(42); the matrices $\Pi(\breve{\theta}_s)$, $M(\breve{\theta}_s,\psi)$ and $V(\breve{\theta}_s,\psi)$ are defined by

$$\Pi(\breve{\theta}_s){:=}P(\breve{\theta}_s)\begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1}\end{pmatrix}P^{-1}(\breve{\theta}_s) \quad , \quad M(\breve{\theta}_s,\psi){:=}P(\breve{\theta}_s)\begin{pmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Psi\end{pmatrix}P^{-1}(\breve{\theta}_s)$$

$$V(\breve{\theta}_s,\psi){:=}\Theta(\breve{\theta}_s) - \Xi(\breve{\theta}_s)\Gamma_f(\breve{\theta}_s)M(\breve{\theta}_s,\psi)$$

where $\Theta{:=}(\Gamma_0^{\Xi} - \Gamma_f\Phi_{c,1})$; Ψ is a $n_2 \times n_2$ matrix containing arbitrary auxiliary parameters unrelated to $\breve{\theta}_s$, and the non-singular $n\times n$ matrix $P(\breve{\theta}_s)$ is obtained from the Jordan normal form of $G(\breve{\theta}_s)$:

$$G(\breve{\theta}_s){:=}P(\breve{\theta}_s)\begin{pmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2\end{pmatrix}P^{-1}(\breve{\theta}_s)$$

where Λ_1 is the $n_1\times n_1$ Jordan normal block that collects the eigenvalues of $G(\breve{\theta}_s)$ that lie inside the unit disk and Λ_2 is the $n_2\times n_2$ Jordan normal block that collects the eigenvalues of $G(\breve{\theta}_s)$ that lie outside the unit disk.

Proof (heuristic). We first re-write system (39) with $\mathring{C}_f$ and $\mathring{C}_0$ given in eq.s (45)-(46), obtaining

$$\mathring{X}_{F,t} = (I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1}\mathring{D}E_t\mathring{X}_{F,t+1} + (I_{2n} - \mathring{D}\mathring{\Phi}_c)^{-1}\mathring{w}_t.$$

This system has the block structure

$$\begin{pmatrix} X^G_{F,t} \\ 0_{n\times 1} \end{pmatrix} = \begin{pmatrix} G(\breve{\theta}_s) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix} \begin{pmatrix} X^G_{F,t-1} \\ 0_{n\times 1} \end{pmatrix} + \begin{pmatrix} \Theta^{-1} & (\Theta^{-1} - \Gamma_f\Phi_{c,2}) \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix} \begin{pmatrix} \varepsilon^\Xi_t \\ 0_{n\times 1} \end{pmatrix}$$

so that the condition $\lambda_{\max}(G(\breve{\theta}_s)){>}1$ implies $\lambda_{\max}(\mathring{C}_f){>}1$. The solution is therefore determined by the solution to the sub-system

$$X^G_{F,t} = G(\breve{\theta}_s)E_tX^G_{F,t+1} + \Theta^{-1}\varepsilon^\Xi_t. \tag{51}$$

We now consider the Jordan normal form of the matrix $G(\breve{\theta}_s)$:

$$G(\breve{\theta}_s){:=}P\begin{pmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{pmatrix}P^{-1} \tag{52}$$

where $P{:=}P(\breve{\theta}_s)$ is non-singular, Λ_1 is the normal Jordan block that collects the $n_1{:=}n - n_2$ eigenvalues of $G(\breve{\theta}_s)$ that lie inside the unit circle, and Λ_2 is the normal Jordan block that collects the eigenvalues that lies outside the unit circle. Using eq.(52), system (51) reads

$$X^G_{F,t} = P\begin{pmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{pmatrix}P^{-1}E_tX^G_{F,t+1} + \Theta^{-1}\varepsilon^\Xi_t$$

and can be transformed into

$$P^{-1}X^G_{F,t} = \begin{pmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{pmatrix}P^{-1}E_tX^G_{F,t} + P^{-1}\Theta^{-1}\varepsilon^\Xi_t$$

and finally partitioned in the form

$$\begin{matrix} n_1\times 1 \\ n_2\times 1 \end{matrix} \begin{pmatrix} X^{G_1}_{F,t} \\ X^{G_2}_{F,t} \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{pmatrix} E_t \begin{pmatrix} X^{G_1}_{F,t+1} \\ X^{G_2}_{F,t+1} \end{pmatrix} + \begin{pmatrix} \vartheta^{G_1}_t \\ \vartheta^{G_2}_t \end{pmatrix} \tag{53}$$

where

$$\begin{pmatrix} X^{G_1}_{F,t} \\ X^{G_2}_{F,t} \end{pmatrix}{:=}P^{-1}X^G_{F,t} \quad , \quad \begin{pmatrix} \vartheta^{G_1}_t \\ \vartheta^{G_2}_t \end{pmatrix}{:=}P^{-1}\Theta^{-1}\varepsilon^\Xi_t. \tag{54}$$

Note that the term on the right in eq. (54) is a MDS with respect to $\mathcal{F}_t$ like ε^Ξ_t. The first block of n_1 equations of system (53) is given by

$$X^{G_1}_{F,t} = \Lambda_1E_tX^{G_1}_{F,t+1} + \vartheta^{G_1}_t \tag{55}$$

and can be regarded as a special case of the multivariate Cagan model of system (39) ($\mathring{C}_f$:=Λ_1, $\mathring{C}_0$:=I_{n_1}). Moreover, since Λ_1 contains only stable eigenvalues, the solution to sub-system (55) is given by

$$X_{F,t}^{G_1} = \vartheta_t^{G1}. \tag{56}$$

The second block of n_2 equations of system (53) is given by

$$X_{F,t}^{G_2} = \Lambda_2 E_t X_{F,t+1}^{G_2} + \vartheta_t^{G2} \tag{57}$$

and, given the non-singularity of Λ_2, can be re-written in the form

$$X_{F,t+1}^{G_2} = \Lambda_2^{-1} X_{F,t}^{G_2} - \Lambda_2^{-1}\vartheta_t^{G2} + \eta_{2,t+1}$$

where we have used the decomposition $\eta_{2,t}$:=$X_{F,t+1}^{G_2} - E_t X_{F,t+1}^{G_2}$. Since both $\eta_{2,t}$ and ϑ_t^{G2} are MDS with respect to $\mathcal{F}_t$, the linear relationship between these two components can be specified in the form

$$\eta_{2,t} = \Psi\vartheta_t^{G_2} + s_t \tag{58}$$

where Ψ is an $n_2 \times n_2$ matrix of arbitrary auxiliary parameters, i.e., unrelated to θ, and s_t is a MDS with respect to $\mathcal{F}_t$ which can be orthogonal to $\vartheta_t^{G_2}$. By substituting eq.(58) in eq.(57) and lagging variables, the system reads as a stable VARMA(1,1)-type process:

$$X_{F,t}^{G_2} = \Lambda_2^{-1} X_{F,t-1}^{G_2} - \Lambda_2^{-1}\vartheta_{t-1}^{G_2} + \Psi\vartheta_t^{G_2} + s_t \tag{59}$$

By coupling sub-systems (56) and (59), the solution is given by

$$\begin{aligned} X_{F,t}^{G_1} &= \vartheta_t^{G_1} \\ X_{F,t}^{G_2} &= \Lambda_2^{-1} X_{F,t-1}^{G_2} - \Lambda_2^{-1}\vartheta_{t-1}^{G_2} + \Psi\vartheta_t^{G_2} + s_t \end{aligned}$$

and is equal, using matrix notation, to

$$\begin{pmatrix} X_{F,t}^{G_1} \\ X_{F,t}^{G_2} \end{pmatrix} = \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{pmatrix} \begin{pmatrix} X_{F,t-1}^{G_1} \\ X_{F,t-1}^{G_2} \end{pmatrix} + \begin{pmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Psi \end{pmatrix} \begin{pmatrix} \vartheta_t^{G_1} \\ \vartheta_t^{G_2} \end{pmatrix}$$
$$+ \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{pmatrix} \begin{pmatrix} \vartheta_{t-1}^{G_1} \\ \vartheta_{t-1}^{G_2} \end{pmatrix} + \begin{pmatrix} 0_{n_1\times 1} \\ s_t \end{pmatrix}.$$

Exploiting the mappings in eq.s (54), this system can be also expressed in the form

$$X_{F,t}^{G} = P(\breve{\theta}_s) \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{pmatrix} P(\breve{\theta}_s)^{-1} X_{F,t-1}^{G} + P(\breve{\theta}_s) \begin{pmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Psi \end{pmatrix} P(\breve{\theta}_s)^{-1}\Theta^{-1}\varepsilon_t^{\Xi}$$

$$+\, P(\breve{\theta}_s)\left(\begin{array}{cc} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{array}\right) P(\breve{\theta}_s)^{-1}\Theta^{-1}\varepsilon^{\Xi}_{t-1}+P(\breve{\theta}_s)\zeta_t \quad (60)$$

where $\zeta_t{:=}(0'_{n_1\times 1}, s'_t)$. Eq.(60) can be further simplified by

$$X^G_{F,t} = \Pi(\breve{\theta}_s)\ X^G_{F,t-1} + M(\breve{\theta}_s,\kappa)\Theta^{-1}\varepsilon^{\Xi}_t - \Pi(\breve{\theta}_s)\Theta^{-1}\varepsilon^{\Xi}_{t-1} + P(\breve{\theta}_s)\zeta_t \quad (61)$$

where the matrices $\Pi(\breve{\theta}_s)$ and $V(\breve{\theta}_s,\kappa)$ are defined by

$$\Pi(\breve{\theta}_s){:=}P\left(\begin{array}{cc} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{array}\right)P^{-1} \quad , \quad M(\breve{\theta}_s,\psi){:=}P\left(\begin{array}{cc} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Psi \end{array}\right)P^{-1}. \quad (62)$$

In terms of $\mathring{X}_{F,t}{:=}(X^{G\prime}_{F,t}, 0'_{n\times 1})'$ and $\mathring{\varepsilon}_t{:=}(\varepsilon^{\Xi\prime}_t, 0'_{n\times 1})'$, the solutions in eq.(61) read

$$\left(\begin{array}{c} X^S_{F,t} \\ 0_{n\times 1} \end{array}\right) = \left(\begin{array}{cc} \Pi(\breve{\theta}_s) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{array}\right)\left(\begin{array}{c} X^G_{F,t-1} \\ 0_{n\times 1} \end{array}\right) + \left(\begin{array}{cc} M(\breve{\theta}_s,\kappa)\Theta^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{array}\right)\left(\begin{array}{c} \varepsilon^{\Xi}_t \\ 0_{n\times 1} \end{array}\right)$$

$$-\left(\begin{array}{cc} \Pi(\breve{\theta}_s)\Theta^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{array}\right)\left(\begin{array}{c} \varepsilon^{\Xi}_{t-1} \\ 0_{n\times 1} \end{array}\right) + \left(\begin{array}{c} P(\breve{\theta}_s)\zeta_t \\ 0_{n\times 1} \end{array}\right)$$

and can be compacted in the canonical form

$$\mathring{X}_{F,t} = \mathring{\Pi}\mathring{X}_{F,t-1} + \mathring{M}\mathring{\Xi}\mathring{\varepsilon}_t - \mathring{\Pi}\mathring{\Xi}\mathring{\varepsilon}_{t-1} + \mathring{\xi}_t \quad (63)$$

where $\mathring{\xi}_t{:=}(\zeta'_t P', 0'_{n\times 1})'$ and

$$\mathring{\Pi}{:=}\left(\begin{array}{cc} \Pi(\breve{\theta}_s) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{array}\right) \quad , \quad \mathring{M}{:=}\left(\begin{array}{cc} M(\breve{\theta}_s,\psi) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{array}\right) \quad , \quad \mathring{\Xi}{:=}\left(\begin{array}{cc} \Theta^{-1}(\breve{\theta}_s) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{array}\right).$$

By combining eq.(63) with eq.(38), the solutions can be compacted in the expression

$$\mathring{X}_t \ = (\mathring{\Pi} + \mathring{\Phi})\mathring{X}_{t-1} - \mathring{\Pi}\ \mathring{\Phi}\mathring{X}_{t-2} + \ \mathring{M}\mathring{\Xi}\ \mathring{\varepsilon}_t - \mathring{\Pi}\mathring{\Xi}\ \mathring{\varepsilon}_{t-1} + \mathring{\xi}_t \quad (64)$$

which, using the lag operator, is equivalent to

$$(I_{2n} - \mathring{\Pi}L)(I_{2n} - \mathring{\Phi}L)\mathring{X}_t = (\mathring{M} - \mathring{\Pi}L)\mathring{\Xi}\ \mathring{\varepsilon}_t + \mathring{\xi}_t. \quad (65)$$

The sub-system delimited by the first n equations of system (65) is given by

$$(I_n - \Pi(\breve{\theta}_s)L)(I_n - \Phi_1(\breve{\theta}_s)L - \Phi_2(\breve{\theta}_s)L^2)X_t = (M(\breve{\theta}_s,\kappa) - \Pi(\breve{\theta}_s)L)\Theta^{-1}\varepsilon^{\Xi}_t + P(\breve{\theta}_s)\zeta_t. \quad (66)$$

Since in this model the forecast error $\eta_t{:=}X_t - E_{t-1}X_t$ has structure

$$\eta_t{:=}X_t - E_{t-1}X_t{:=}M\Theta^{-1}\varepsilon_t^R + P\zeta_t,$$

from the definition $\varepsilon_t^{\Xi}{:=}\varepsilon_t + \Xi\Gamma_f\eta_t$ and the assumption that all the eigenvalues of the matrix $\Xi(\breve{\theta}_s)\Gamma_f(\breve{\theta}_s)M(\breve{\theta}_s,\psi)\Theta(\breve{\theta}_s)^{-1}$ are different from 1, it is possible to obtain the relationship

$$\varepsilon_t^{\Xi}{:=}(I_n - \Xi\Gamma_f M\Theta^{-1})^{-1}(\varepsilon_t + \Xi\Gamma_f\tau_t)$$
$$=(I_n - \Xi\Gamma_f M\Theta^{-1})^{-1}\varepsilon_t + (I_n - \Xi\Gamma_f M\Theta^{-1})^{-1}\Xi\Gamma_f P\zeta_t.$$

If this expression is substituted into the right-hand side of eq. (66), rearranging terms and using the definitions $V(\breve{\theta}_s,\psi){:=}\Theta(\breve{\theta}_s){-}\Xi(\breve{\theta}_s)\Gamma_f(\breve{\theta}_s)M(\breve{\theta}_s,\psi)$ and $\tau_t{:=}[M(\breve{\theta}_s,\psi){-}\Pi(\breve{\theta}_s)L]V(\breve{\theta}_s,\psi)^{-1}P(\breve{\theta}_s)\zeta_t{+}P(\breve{\theta}_s)\zeta_t$, we obtain the representation in eq.s (49)-(50). ■

Corollary 1 [Necessary condition for a VARMA-type reduced form solution] Consider the new-Keynesian system (32)-(33) and Assumptions 1-2. Assume that all stable linear reduced form solutions of interest are given either by the VAR system (47) or by the VARMA-type system (49)-(50), respectively. If for a given $\theta_s{=}\breve{\theta}_s$ the data generating process belongs to the class of VARMA-type reduced forms in Eqs. (49)-(50) and Minimum State Variable (MSV) solutions are ruled out, then $\lambda_{\max}(G(\breve{\theta}_s)){>}1$.

Proof. Proposition 1 establishes that for $\theta_s{=}\breve{\theta}_s$, the condition $\lambda_{\max}(G(\breve{\theta}_s)){<}1$ is sufficient for the existence of the finite-order VAR representation in eq.(47). By negation, any non-MSV reduced form solution described by the class of models in eq.s (49)-(50) must satisfy, for $\theta_s{=}\breve{\theta}_s$, the condition $\lambda_{\max}(G(\breve{\theta}_s)){>}1$. ■

We remark that $\lambda_{\max}(G(\breve{\theta}_s)){<}1$ is not necessary for the existence of the finite-order VAR representation in eq. (47). To see this, it is sufficient to observe the MSV solution nested within system (49)-(50) for $\Psi{=}I_{n_2}$ and $s_t{=}0_{n_2\times 1}$ a.s. $\forall t$, collapses to system (47) but is such that $\lambda_{\max}(G(\theta)){>}1$.

Asymptotic properties of the testing strategy

In this section, we formalize some asymptotic properties of the '$LR_T \rightarrow AR_T$' testing strategy discussed in Sub-section 3.3 of the paper.

For convenience, the hypotheses H_0' and $H_{0,cer}$ in, respectively, eq. (15) and eq. (20) of the paper, are reported below:

$$H_0' : X_t \text{ is generated by the VAR system (47) under the CER in eq. (48);} \quad (67)$$

$$\text{there exists } \theta_\varepsilon \text{ such that } H_{0,cer}\text{: } \phi_{\breve{\theta}_s}{=}g(\breve{\theta}_s, \theta_\varepsilon) \ \ , \ \ \theta_s = \breve{\theta}_s \in \mathcal{P}_{\theta_s}. \quad (68)$$

$H_{0,cer}$ is the composite null hypothesis that the CER implied by the new-Keynesian system are valid for a given $\theta_s{=}\breve{\theta}_s$. We recall that H_0' is accepted if there exists at least one $\theta_s = \breve{\theta}_s$ such that $H_{0,cer}$ is not rejected; instead, H_0' is rejected only if $H_{0,cer}$ is rejected for all values of the parameters. The alternative hypothesis of multiple equilibria, H_1' (eq. (16) of the paper), is also reported here:

$$H_1' : X_t \text{ is generated by the VARMA-type system (49)-(50)} \quad (69)$$

where $\theta^* \in \mathcal{I}^0$ and:

$$\mathcal{I}^0{:=}\left\{\theta^*{:=}(\theta', \psi', \sigma_\zeta^{+\prime})', \, \theta_s \in \mathcal{P}_{\theta_s}^I, \, \psi \in \mathcal{N} \setminus \left\{vec(I_{(n_2)^2})\right\}, \, \sigma_\zeta^+ \in \mathcal{Z}\setminus\left\{0_{6\times 1}\right\}\right\} \subset \mathcal{I}, \quad (70)$$

$$\mathcal{I}{:=}\left\{\theta^*{:=}(\theta', \psi', \sigma_\zeta^{+\prime})', \, \theta_s \in \mathcal{P}_{\theta_s}^I, \, \psi \in \mathcal{N}, \, \sigma_\zeta^+ \in \mathcal{Z}\right\}.$$

It can be noticed that the alternative H_1' is specified such that MSV equilibria are ruled out, see Section 2 of the paper. The logic upon which the '$LR_T \to AR_T$' approach is based is summarized in Table TS1.

By construction, the size of our testing strategy, i.e. the probability of rejecting H_0' when H_0' is true, depends on the test $LR_T(\hat{\phi}_{\theta_s})$ computed in the first-step. Let $P_{\theta_s,T}^{LR}[\cdot]$ be the probability measure associated with the distribution of the $LR_T(\hat{\phi}_{\theta_s})$ test in a sample of length T. The notation '$P_{\theta_s,T}^{LR}[\cdot]$' remarks that in small samples the distribution of $LR_T(\hat{\phi}_{\theta_s})$ generally depends on θ_s. However, under $H_{0,cer}$, the asymptotic null distribution of the test is pivotal and is $\chi_{d_1}^2$ with $d_1{:=}\dim(\phi) - \dim(\theta_\varepsilon)$, regardless of whether θ_s is identified or not, see e.g. Guerron-Quintana *et al.* (2013). Therefore, defined the size of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test for the hypothesis $H_{0,cer}$ in a sample of length T by

$$\eta_{1,T}^0{:=}\sup_{\breve{\theta}_s \in \mathcal{P}_{\theta_s}} P_{\breve{\theta}_s,T}^{LR}[LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}], \quad (71)$$

where $c_T^{\eta_1}$ is the critical value of the test at the nominal level $0{<}\eta_1{<}1$,[17] for $\breve{\theta}_s = \theta_{0,s}$ it holds

$$\eta_{1,\infty}^0{:=}\limsup_{T\to\infty} \eta_{1,T}^0 = \eta_1 \quad (72)$$

[17] In the definition of the size $\eta_{1,T}^0$ in eq. (71), we have not restricted the parameter space to the determinacy region, because the hypothesis $H_{0,cer}$ may also hold for points that lie in the indeterminacy region and for which MSV solutions occur. We thank a referee for bringing this point to our attention.

which implies that the test $LR_T(\hat{\phi}_{\breve{\theta}_s})$ has correct asymptotic size for $H_{0,cer}$. This ensures that the identification-robust confidence set reported in eq. (21) of the paper has asymptotic coverage 1$-$ η_1.

However, the hypothesis we are actually interested in is H_0'. H_0' is rejected if and only if there is no $\theta_s = \breve{\theta}_s$ for which the test $LR_T(\hat{\phi}_{\breve{\theta}_s})$ accepts the CER. Therefore, the asymptotic size of the test for H_0' is given by

$$\eta_{1,\infty}{:=}\limsup_{T\to\infty}\ \eta_{1,T} \quad , \quad \eta_{1,T}{:=}P^{LR}_{\breve{\theta}_s,T}\left[\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}\right]. \tag{73}$$

The next proposition establishes that the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test is asymptotically correct for H_0', with asymptotic size at most η_1.

Proposition 3 [Asymptotic size for H_0'] Consider the new-Keynesian system in eq.s (32)-(33), and the '$LR_T \to AR_T$' testing strategy summarized in Sub-section 3.3 of the paper. Let $\theta_0{:=}(\theta_{0,s}', \theta_{0,\varepsilon})' \in \mathcal{P}$ be the true value of θ. Under H_0' and for $\breve{\theta}_s{=}\theta_{0,s}$, the $LR_T(\hat{\phi}_{\theta_{0,s}})$ test is such that $\eta_{1,\infty} \leq \eta_1$, where η_1 is a pre-fixed type-I error.

Proof Since $P^{LR}_{\breve{\theta}_s,T}\left[\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}\right] \leq P^{LR}_{\breve{\theta}_s,T}\left[LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}\right]$, for $\breve{\theta}_s{=}\theta_{0,s}$ we have

$$P^{LR}_{\breve{\theta}_{0,s},T}\left[\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}\right] \leq \eta^0_{1,T} \tag{74}$$

Taking the limsup and using eq. (72) we obtain the result.■

Proposition 3 ensures that the identification-robust confidence set reported in eq. (30) of the paper:

$$\mathcal{C}^{*LR}_{1-\eta_1}{:=}\left\{\breve{\theta}_s \in \mathcal{G}^*_{\theta_s},\ LR_T(\hat{\phi}_{\breve{\theta}_s}) < c^{\eta_1}_{\chi^2_{d_1}}\right\}$$

has asymptotic coverage at least $1-\eta_1$.

When the null H_0' is rejected, a second-step is run to decide whether the alternative hypothesis of multiple equilibria H_1' must be accepted, or rejected. If also H_1' is rejected, we conclude that the specified system of Euler structural equations omits important propagation mechanisms. The second-step of the '$LR_T \to AR_T$' procedure is based on the test $AR_T(\breve{\theta}_s)$ for the hypothesis $H_{0,spec}$ in eq. (23) of the paper, here reported for convenience:

$$H_{0,spec}\colon\ \theta_s{=}\breve{\theta}_s \quad ,\ \breve{\theta}_s \in \mathcal{P}_{\theta_s}. \tag{75}$$

The hypothesis H_1' is accepted if there exists at least one $\theta_s{=}\breve{\theta}_s$ such that $H_{0,spec}$ is not rejected; instead, H_1' is rejected if and only if $H_{0,spec}$ is rejected for all values of the parameters. Using the same arguments we have used for the test $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test in Proposition 3 and exploiting the results in Dufour *et al.* (2006, 2009, 2010, 2013), it is possible to conclude that asymptotically, the probability of incorrectly rejecting the hypothesis H_1' by the $AR_T(\breve{\theta}_s)$ test is bounded above by the nominal type-I error pre-fixed in the second-step, η_2. Thus, the identification-robust confidence set reported in eq. (31) of the paper,

$$\mathcal{C}^{*AR}_{1-\eta_2}{:=}\left\{\breve{\theta}_s \in \mathcal{D}^*_{\theta_s},\ AR_T(\breve{\theta}_s) < c^{\eta_2}_{\chi^2_{d_2}}\right\}$$

where $\mathcal{D}^*_{\theta_s}{:=}\left\{\breve{\theta}_s \in \mathcal{P}_{\theta_s},\ \lambda_{\max}(G(\breve{\theta}_s)){>}1\right\}$, has asymptotic coverage at least $1-\eta_2$.

Finally, as observed in Remark 4 of the paper (Sub-section 3.3), the hypothesis of no dynamic misspecification of the new-Keynesian system is given by $H^* = H_0'\vee H_1'$. The sequence of tests $LR_T(\hat{\phi}_{\breve{\theta}_s})$ and $AR_T(\breve{\theta}_s)$ can be used as a (mis)specification test for H^*: when the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test rejects H_0' in the first-step and the $AR_T(\breve{\theta}_s)$ test rejects H_1' in the second-step, the new-Keynesian model is rejected; the new-Keynesian model is instead accepted either when the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test accepts H_0' in the first-step, or when the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test rejects H_0' in the first-step but the $AR_T(\breve{\theta}_s)$ test accepts H_1' in the first step. The next proposition establishes that, asymptotically, the probability that the '$LR_T \to AR_T$' testing strategy incorrectly rejects H^* is bounded above by the maximum of the nominal type-I errors η_1 and η_2 pre-fixed for the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test and the $AR_T(\breve{\theta}_s)$ test, respectively.

Proposition 4 [Asymptotic size for the null of no dynamic misspecification] Consider the new-Keynesian system in eq.s (32)-(33), and the '$LR_T \to AR_T$' testing strategy summarized in Sub-section 3.3 of the paper. Let $\theta_0{:=}(\theta_{0,s}', \theta_{0,\varepsilon})' \in \mathcal{P}$ be the true value of θ. Under $H^* = H_0'\vee H_1'$ and for $\breve{\theta}_s{=}\theta_{0,s}$, the asymptotic probability of incorrectly rejecting H^* is bounded above by $\max\{\eta_1\ ,\ \eta_2\}$.

Proof Let $R_{1,T}{:=}\left\{\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}\right\}$ and $R_{2,T}{:=}\left\{\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} AR_T(\breve{\theta}_s) \geq c_T^{\eta_2}\right\}$.

From Proposition 3 we know that for $\breve{\theta}_s{=}\theta_{0,s}$,

$$\Pr(R_{1,T} \mid H_0'){:=}\limsup_{T\to\infty} P^{LR}_{\theta_{0,s},T}\left[\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c_T^{\eta_1}\right] \leq \eta_1.$$

Similarly,

$$\Pr(R_{2,T} \mid H_1'):=\limsup_{T\to\infty} P^{LR}_{\theta_{0,s},T}\left[\min_{\breve{\theta}_s\in\mathcal{P}_{\theta_s}} AR_T(\breve{\theta}_s) \geq c_T^{\eta_2}\right] \leq \eta_2.$$

Then

$$\Pr(\text{reject } H^* \mid H^*)=\Pr(R_{1,T}\wedge R_{2,T} \mid H^*) \leq \max\left\{ \begin{array}{c} \Pr(R_{1,T}\wedge R_{2,T} \mid H_0') \ , \\ \Pr(R_{1,T}\wedge R_{2,T} \mid H_1') \end{array} \right\}$$

$$\leq \max\left\{\Pr(R_{1,T} \mid H_0') \ , \ \Pr(R_{2,T} \mid H_1')\right\} \leq \max\left\{\eta_1 \ , \eta_2\right\}. \blacksquare$$

Further Monte Carlo results

In this section we complete the Monte Carlo experimentation provided in the paper, by discussing further results on the finite sample properties of the '$LR_T \to AR_T$' testing strategy. In Sub-section .1 we consider the power of the test against some specific non-MSV equilibria that belong to the class of time series model in eq.s (49)-(50) (see also the hypothesis H_1' in eq. (16) of the paper and Section). In Sub-section .2 we investigate the power of the test against the hypothesis of omission of relevant propagation mechanisms from the specified system of Euler equations.

Before discussing the finite sample power of the '$LR_T \to AR_T$' testing strategy, it is worth coming back on the results reported in Table 1 of the paper about empirical size. In the first-step, we build the identification-robust confidence set $\mathcal{C}^{*LR}_{1-\eta_1}:=\left\{\breve{\theta}_s \in \mathcal{G}^*_{\theta_s},\ LR_T(\hat{\phi}_{\breve{\theta}_s}) < c^{\eta_1}_{\chi^2_{d_1}}\right\}$, where $\mathcal{G}^*_{\theta_s}$ is the grid used to invert the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test for the CER. The hypothesis H_0' is rejected at the pre-fixed level η_1 if $\mathcal{C}^{*LR}_{1-\eta_1}$ is empty, and it is accepted otherwise. Obviously, $\mathcal{C}^{*LR}_{1-\eta_1}$ is empty when $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}}):=\min_{\theta_s\in\mathcal{G}^*_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \geq c^{\eta_1}_{\chi^2_{d_1}}$. It turns out that under the assumption of correct specification, which in our case includes the hypothesis that the chosen parametric grid $\mathcal{G}^*_{\theta_s}$ contains the true value $\theta_{s,0}$, the following inequality holds:

$$LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}}):=\min_{\breve{\theta}_s\in\mathcal{G}^*_{\theta_s}} LR_T(\hat{\phi}_{\breve{\theta}_s}) \leq \ LR_T(\hat{\phi}_{\theta_{s,0}}).$$

This inequality suggests that whatever the method one uses to invert the test, the empirical rejection frequency associated with $LR_T(\hat{\phi}_{\theta_{s,0}})$, which is the test statistic for the hypothesis $H_{0,cer}$ in eq. (20) of the paper evaluated at the specific point $\theta_s = \theta_{s,0}$, is an upper bound for the size of the grid-testing procedure for H_0'. Thus, Table 1 of the paper also reports the empirical size of the test $LR_T(\hat{\phi}_{\theta_{0,s}})$, other than the empirical size for the hypothesis of interest, H_0'.

.1 Power against indeterminate equilibria

We recall that the '$LR_T \rightarrow AR_T$' testing strategy rejects the null H_0' (see eq. (15) in the paper) when the LR test computed in the first-step rejects the CER implied by the new-Keynesian model under determinacy.

The data generating processes used in this experiment are selected from the VARMA-type reduced form solutions in eq.s (49)-(50) for specific values of the structural and auxiliary parameters. In this case, we can only provide limited Monte Carlo experimentation because, given the structural parameters and the fundamental shocks, the choice of ψ and σ_ζ^+ is completely arbitrary. To simplify the analysis, we follow Lubik and Schorfheide (2004) and Fanelli (2012) and focus on the situation in which the sunspot shocks are set to zero, i.e. $\sigma_\zeta^+ := 0_{6\times 1}$ ($\Rightarrow \tau_t := 0_{3\times 1}$ a.s. $\forall\ t$) in eq.s (49)-(50). This scenario is often referred to as 'indeterminacy without sunspots'. The rejection frequency of the testing strategy is expected to increase when also the sunspot shocks are allowed to affect the dynamics of the system.

The vector $\theta_{0,s} := (\theta_{0,s}', \theta_{0,\varepsilon}')'$ is calibrated at the medians of the 90% coverage percentiles of the posterior distribution reported in Table 1 of Benati and Surico (2009), 'Before October 1979' column. With this choice, the largest eigenvalue of the matrix $G(\theta_{0,s})$ is equal to $\lambda_{\max}(G(\theta_{0,s}))=1.0051$ and only one eigenvalue lies outside the unit circle, so that ψ is scalar. We consider three possible values for ψ: 0.95, 1.05 and 0.5. The choices $\psi=0.95$ and $\psi=1.05$ are deliberately close to the case $\psi=1$ that generates MSV solutions observationally equivalent to the unique stable equilibrium, see Sub-section 2.3 of our paper.[18]

Artificial datasets of length $T=100$ are generated from system (49)-(50) which, after the qualifications discussed above, reads as a 'pure' VARMA(3,1) system with highly restricted parameters. In this experiment, it is also interesting to investigate the empirical size of the test $AR_T(\breve{\theta}_s)$ computed in the second-step of the testing strategy. The test $AR_T(\breve{\theta}_s)$ is by construction robust to determinacy/indeterminacy, hence we can check whether the empirical size of this test is confined to admissible levels for the specific DGPs under scrutiny. We fix the nominal type-I errors of the two tests, η_1 and η_2, at the level $\eta_1 = \eta_2 := 0.10$.

[18] Aside from these finite sample simulations, we deliberately ignore testing issues at the boundary of H_1' and H_0'.

The results are summarized in Table TS2. We observe that the rejection frequency of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test for the CER computed in the first-step is reasonably good even when the specified indeterminate reduced form solution is close to the MSV solution: the empirical power is 67.5% for ψ=0.95 and 76.9% for ψ=1.05, and is 100% for the selected indeterminate equilibrium more distant from the MSV solution (ψ=0.50). The rejection frequency of the $AR_T(\breve{\theta}_s)$ test, instead, is to some extent influences by the value taken by the nuisance parameter ψ which, recall, may amplify or dampen the oscillations of the variables in X_t through the moving average part of system (49), in addition to what implied by the fundamental shocks. In samples of size T=100, the empirical size of the computed $AR_T(\breve{\theta}_s)$ test ranges from 7.3% for ψ=0.95 to 12.5% for ψ=0.50, and is equal to 8% for ψ=1.05, as opposed to the pre-fixed type-I error of 10%. We can conclude that the under(over)-rejection phenomenon is confined to admissible levels.

.2 Power against the omission of propagation mechanisms

When the hypothesis H_0' (see eq. (15) in the paper) is rejected in the first-step, and the hypothesis H_1' (see eq. (16) in the paper) is rejected in the second-step, the '$LR_T \rightarrow AR_T$' testing strategy leads one to conclude that the specified system of Euler equations is 'dynamically misspecified' in the sense that it omits important propagation mechanisms. This situation occurs when the two identification-robust confidence sets in eq. (30) and eq. (31) of the paper are empty. In this sub-section, we analyze the rejection frequency of the testing strategy in these situations.

The data generating process is assumed to belong to the reduced form solutions associated with the 'augmented' system of Euler equations:

$$\Gamma_0^{\Xi} X_t = \Gamma_f E_t X_{t+1} + \sum_{h=2}^{k_2} \Gamma_{f,h} E_t X_{t+h} + \Gamma_{b,1}^{\Xi} X_{t-1} + \Gamma_{b,2}^{\Xi} X_{t-2} + \sum_{j=3}^{k_1} \Gamma_{b,j} X_{t-j} + \varepsilon_t^R \tag{76}$$

which, compared to the baseline system (34), includes (k_1-2) additional lags of X_t associated with the matrices of parameters $\Gamma_{b,j} \neq 0_{n\times n}$, $j = 3, .., k_1$, $(k_1 \geq 3)$, and $(k_2 - 1)$ additional expectations terms associated with the matrices of parameters $\Gamma_{f,h} \neq 0_{n\times n}$, $h = 2, .., k_2$, $(k_2 \geq 2)$. All reduced form models discussed in Section are (non-locally) misspecified if at least one among $\Gamma_{b,j}$, $j = 3, .., k_1$ and $\Gamma_{f,h}$, $h = 2, .., k_2$ is different from zero, and the data generating process belongs to the class of reduced form solutions generated by system (76).

We confine our attention to a version of system (76) where the matrices Γ_0, Γ_f, Γ_b and Ξ have the same structure as in Section 2 of the paper, and $\Gamma_{f,h}$:=$0_{3\times 3}$ for $h \geq 2$ and k_1:=3,with $\Gamma_{b,3}$:=μI_3. With this design, the (scalar) parameter μ captures the extent of the (non-local) misspecification of the theoretical model. The 'extended' vector of parameters is given by θ^e:=$(\theta_s', \mu, \theta_\varepsilon')'$. When $\mu = 0$, no dynamic misspecification occurs in the sense that the system (76) collapses to the baseline new-Keynesian model summarized in eq. (34). Conversely, values of μ different from zero and for which a reduced form solution to system (76) exists, define a data generating process for which the '$LR_T \rightarrow AR_T$' testing strategy based on system (34) leads one to reject both H_0' and H_1'.

Artificial samples of length $T = 100$ (not including initial lags) are generated from system (76) under determinacy, by calibrating θ_s as in Table 1 of the paper, and setting the extra parameter μ to values for which a finite-order VAR solution exists. We consider $M = 1,000$ replications. The identification-robust '$LR_T \rightarrow AR_T$' procedure is applied on each simulated sample, using $\eta_1 = \eta_2 = 0.10$ as nominal type-I errors of the two tests. Results are reported in Table TS3 which summarizes the marginal rejection frequencies of the tests $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ (see eq. (22) of the paper) and $AR_T(\hat{\theta}_{s,LI})$ (see eq. (29) of the paper), and their joint rejection frequency.

Table TS3 shows that the rejection frequency of the testing strategy tends to increase, as expected, as the magnitude of the misspecification parameter $|\mu|$ increases. The marginal rejection frequency of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test for H_0' ranges from 58.9% for $\mu = -0.10$, to 100% for $\mu = -0.35$, and is equal to 84.4% and 98.8% for $\mu = -0.15$ and $\mu = -0.25$, respectively. Therefore the risk of falsely accepting a reduced form solution with the same time series representation as the determinate equilibrium in a dynamically misspecified model is under strict control. The marginal rejection frequency of the test $AR_T(\breve{\theta}_s)$ for H_1' ranges from 54.4% for $\mu = -0.10$, to 88.9% for $\mu = -0.35$, and is equal to 63.9% and 80.9% for $\mu = -0.15$ and $\mu = -0.25$, respectively. We notice that the marginal rejection frequency of the $LR_T(\hat{\phi}_{\breve{\theta}_s})$ test is systematically larger than the marginal rejection frequency of the $AR_T(\breve{\theta}_s)$ test, confirming West's (1986) findings that when linear rational expectations models are misspecified, 'full-information' tests tend to be more powerful than 'limited-information' tests.

The joint rejection frequency ranges from 40.6% for $\mu = -0.10$, to 0.88.9% for $\mu = -0.35$, and is equal to 58.9% and 80.4% for $\mu = -0.15$ and $\mu = -0.25$, respectively. Overall, the results in Table TS3 suggest that the capacity of the '$LR_T \rightarrow AR_T$' testing strategy to reject the hypotheses H_0' and H_1' when the NK model omits important propagation mechanisms is satisfactory.

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TABLES

Table TS1. Summary of the '$LR_T \rightarrow AR_T$' testing strategy for the new-Keynesian system (32)-(33).

<table>
<tr><td colspan="3">Step 1: $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ test rejects the CER ($\mathcal{C}^{*LR}_{1-\eta_1}$ empty) ?</td></tr>
<tr><td colspan="2">YES</td><td>NO</td></tr>
<tr><td colspan="2">Step 2: $AR_T(\hat{\theta}_{s,LI})$ test rejects the OR ($\mathcal{C}^{*AR}_{1-\eta_2}$ empty) ?</td><td></td></tr>
<tr><td>YES</td><td>NO</td><td>H_0' in eq. (67) accepted</td></tr>
<tr><td>Omission of
propagation
mechanisms</td><td>H_1' in eq. (69) accepted
Indeterminacy</td><td>Non conclusive evidence of determ.
(sunspot shocks and param. indet. ruled out)</td></tr>
<tr><td>new-Keynesian model rejected</td><td>new-Keynesian model accepted</td><td>new-Keynesian model accepted</td></tr>
</table>

Table TS2. Empirical power of the '$LR_T \rightarrow AR_T$' testing strategy against indeterminate equilibria and empirical size of the $AR_T(\breve{\theta}_s)$ test in the second-step.

true $\theta_{0,s}$: $\lambda_{\max}(G(\theta_{0,s}))$:=1.0051		T=100	η_1=η_2=0.10
γ_0:=0.744	Indeterminacy param. :	ψ=0.95	
δ_0:=0.124			Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.675
α_0:=0.059			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.073
κ_0:=0.044			
ρ_0:=0.595			
$\varphi_{\widetilde{y},0}$:=0.527			
$\varphi_{\pi,0}$:=0.821		ψ=1.05	
$\rho_{\widetilde{y},0}$:=0.796			Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.769
$\rho_{\pi,0}$:=0.418			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.08
$\rho_{R,0}$:=0.404			
		ψ=0.50	Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=1
			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.125

NOTES. Results are obtained using M=1,000 replications. Each simulated sample is initiated with 200 additional observations to get a stochastic initial state and then are discarded. The data are generated from the new-Keynesian system (32)-(33) under indeterminacy, see eq.s (49)-(50), assuming that the sunspot shocks are absent, i.e. $\sigma_{\zeta}^{+}= 0$. The structural parameters are calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico (2009), column 'Before October 1979'. The numerical inversions of the tests in the two steps of the procedure are obtained on each generated dataset by drawing 300 points $\breve{\theta}_s$ from the same grid and method as in Table 1 of the paper. ψ is the auxiliary parameter that governs the 'parametric indeterminacy' of the system. The test statistic $AR_T(\hat{\theta}_{s,LI})$ is computed as a quasi-LR test (see Sub-section 3.2 of the paper, eq. (29)), using Z_t:=$(X'_{t-1}, X'_{t-2}, ..., X'_{t-r})'$ and $r = 7$ in the auxiliary multivariate regression. $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ is computed as in eq. (22) of the paper. Rej($\cdot$) stands for 'rejection frequency'.

Table TS3. Empirical power of the ‘$LR_T \rightarrow AR_T$’ testing strategy when the data are generated from the new-Keynesian system (76) with $\Gamma_{f,h}$:=$0_{3\times 3}$ for $h \geq 2$ and k_1:=3,and $\Gamma_{b,3}$:=μI_3.

true $\theta_{0,s}$:			
		T=100	η_1=η_2=0.10
γ_0:=0.744	misspecification param.:	μ=-0.10	
δ_0:=0.124			Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.625
α_0:=0.059			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.568
κ_0:=0.044			Rej($AR_T(\hat{\theta}_{s,LI})$; $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.432
ρ_0:=0.595			
$\varphi_{\widetilde{y},0}$:=0.527		μ=-0.15	
$\varphi_{\pi,0}$:=0.821			Rej($LR_T(\hat{\phi}_{\breve{\theta}_{s,ML}})$)=0.847
$\rho_{\widetilde{y},0}$:=0.796			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.645
$\rho_{\pi,0}$:=0.418			Rej($AR_T(\hat{\theta}_{s,LI})$; $LR_T(\hat{\phi}_{\breve{\theta}_{s,ML}})$)=0.593
$\rho_{R,0}$:=0.404			
		μ=-0.25	
			Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.985
			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.810
			Rej($AR_T(\hat{\theta}_{s,LI})$; $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.804
		μ:=-0.35	
			Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=1
			Rej($AR_T(\hat{\theta}_{s,LI})$)=0.913
			Rej($AR_T(\hat{\theta}_{s,LI})$; $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$)=0.913

NOTES. Results are obtained using M=1,000 replications. Size of the burn-in: 200 observations. Artificial datasets are generated from system (76) under determinacy, by calibrating θ_s as in Table 1 of the paper, and setting the extra parameter μ that governs the misspecification of the model to values for which a finite-order VAR solution exists. The numerical inversions of the tests in the two steps of the procedure are obtained on each generated dataset by drawing 300 points $\breve{\theta}_s$ from the same grid and method as in Table 1 of the paper. The test statistic $AR_T(\hat{\theta}_{s,LI})$ is computed as a quasi-LR test (see Sub-section 3.2 of the paper, eq. (29)), using Z_t:=$(X'_{t-1}, X'_{t-2}, ..., X'_{t-r})'$ and $r = 7$ in the auxiliary multivariate regression. $LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$ is computed as in eq. (22) of the paper. Rej($\cdot$) stands for ‘rejection frequency’; Rej($LR_T(\hat{\phi}_{\hat{\theta}_{s,ML}})$; $AR_T(\hat{\theta}_{s,LI})$) denotes the joint rejection frequency.